

Some infinite matrix analysis, a Trotter product formula for dissipative operators, and an algorithm for the incompressible Navier-Stokes equation

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Abstract

We introduce a global scheme on the n -torus of a controlled incompressible Navier-Stokes equation in terms of a coupled controlled infinite ODE-system of Fourier-modes with smooth data. We construct a scheme of global approximations related to linear partial integro-differential equations in dual space which are uniformly bounded in dual Sobolev spaces with polynomially decaying modes. The scheme is based on some infinite matrix algebra related to weakly singular integrals, and a Trotter-product formula for dissipative operators which leads to rigorous existence results and a uniform bound for the solutions of the successive approximating linearized equations in dual space. For data with polynomial decay of the modes (equivalent to smooth data on the n -torus) global existence follows from a compactness property in strong dual Sobolev spaces (spaces of polynomially decaying Fourier modes). The difference to schemes considered in [8, 7, 9, 10, 11] is that we have a priori estimates for solutions of the iterated global linear partial integro-differential equations. Furthermore, the control function does only control the non-dissipative zero modes and is, hence, much simpler. The analysis of the scheme leads to an algorithmic scheme of the Navier-Stokes equation on the torus for regular data (certain polynomially decaying modes) based on the mentioned Trotter-type product formula for specific dissipative operators. This algorithm which is closely related to the Trotter product formula is natural, since the latter is a limit formula with respect to the order of the modes and expresses an approximation of approximating functions via finite systems. It is our main intention here to define an algorithm which uses the damping via dissipative modes and which converges in strong norms for arbitrary viscosity constants $\nu > 0$ (or Reynolds numbers). Finally, we discuss some notions of the concept of 'turbulence' which is not an effect of a blow-up of vorticity, but may be a concept to be described by dynamical properties, as has been suggested by other authors for a long time (cf. [2, 3], and [4]).

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1 Introduction

The formal transformation of partial differential equations to infinite systems of ordinary differential equations via representations in a Fourier basis may be useful if the problem is posed on a torus. Formal representations of solutions are simplified at least if the equations are linear. Well, even non-linear infinite ODE-representations may be useful in order to define solution schemes. However, in order to make sense of an exponential function of an infinite matrix applied to an infinite vector the sequence spaces involved have to be measured in rather strong norms. Schur [16] may have been the first who provided necessary and sufficient conditions such that infinite linear transformations make sense. His criteria were in the sense of l^1 -sequence spaces and are far too weak for our purposes here. However, dealing with the incompressible Navier-Stokes equation we shall assume that the initial vector-valued data function $\mathbf{h} = (h_1, \dots, h_n)^T$ satisfies

$$h_i \in C^\infty(\mathbb{T}^n), \quad (1)$$

where \mathbb{T}^n denotes the n -torus (maybe the only flat compact manifold of practical interest when studying the Navier-Stokes equation). In the following let \mathbb{Z} be the set of integers, \mathbb{R} the field of real numbers, and let $\alpha, \beta \in \mathbb{Z}^n$ be multiindices. Differences of multiindices $\alpha - \beta \in \mathbb{Z}^n$ are built componentwise, and as usual for a multiindex γ with nonnegative entries we denote $|\gamma| = \sum_{i=1}^n |\gamma_i|$. Consider a smooth function $\mathbf{h} = (h_1, \dots, h_n)^T$ with $h_i \in C^\infty(\mathbb{T}_l^n)$, where $\mathbb{T}_l^n = \mathbb{R}^n / l\mathbb{Z}^n$ is the torus of dimension n and size $l > 0$. Note that the smoothness of the functions h_i means that there is a polynomial decay of the Fourier modes, i.e., for a torus of size $l = 1$ for the modes of multivariate differentials of the function h_i we have

$$\begin{aligned} D^\gamma h_{i\alpha} &:= \int_{\mathbb{T}^n} D_x^\gamma h_i(x) \exp(-2\pi i \alpha x) dx \\ &= (-1)^{|\gamma|} \int_{\mathbb{T}^n} h_i(x) \Pi_{i=1}^n (-2\pi i \alpha_i)^{\gamma_i} \exp(2\pi i \alpha x) dx \\ &= \Pi_{i=1}^n (-2\pi i \alpha_i)^{\gamma_i} h_{i\alpha}, \end{aligned} \quad (2)$$

where as usual for a multiindex γ with nonnegative entries $\gamma_i \geq 0$ the symbol D_x^γ denotes the multivariate derivative with respect to x and of order γ_i with respect to the i th component of x . Here $h_{i\alpha}$ is the α th Fourier mode of the function h_i , and $D^\gamma h_{i\alpha}$ is the α th Fourier mode of the function $D_x^\gamma h_i$. Since $D_x^\gamma h_i$ is smooth this function has a Fourier decomposition on \mathbb{T}^n . The sequence $(D^\gamma h_{i\alpha})_{\alpha \in \mathbb{Z}^n}$ of the associated α -modes exists in the space of square integrable sequences $l^2(\mathbb{Z}^n)$. Hence, we have polynomial decay of the modes $h_{i\alpha}$ as $|\alpha| \uparrow \infty$. This polynomial decay is important in the following in order to make sense of certain matrix operations with certain multiindexed infinite matrices and multiplications of these matrices with infinite vectors, because

we are interested in regular solutions. In the following we have in mind that a multiindexed vector

$$\mathbf{u}^F := (u_\alpha)_{\alpha \in \mathbb{Z}^n}^T \quad (3)$$

(the superscript T meaning 'transposed') with complex constants u_α (although we consider only real solutions in this paper) which decay fast enough as $|\alpha| \uparrow \infty$ corresponds to a function

$$u \in C^\infty(\mathbb{T}_l^n), \quad (4)$$

where

$$u(x) := \sum_{\alpha \in \mathbb{Z}^n} u_\alpha \exp\left(\frac{2\pi i \alpha x}{l}\right). \quad (5)$$

If the representation can be rewritten with real functions $\cos\left(\frac{2\pi i \alpha x}{l}\right)$ and $\sin\left(\frac{2\pi i \alpha x}{l}\right)$ and with real coefficients of these functions, then the function in (4) has values in \mathbb{R} , and this is the case we have in mind in this paper. We say that the infinite vector \mathbf{u}^F is in the dual Sobolev space of order $s \in \mathbb{R}$, i.e.,

$$\mathbf{u}^F \in h^s(\mathbb{Z}^n) \quad (6)$$

in symbols, if the corresponding function u defined in (4) is in the Sobolev space $H^s(\mathbb{T}_l^n)$ of order $s \in \mathbb{R}$. We may also define the dual space h^s directly defining

$$\mathbf{u}^F \in h^s(\mathbb{Z}^n) \Leftrightarrow \sum_{\alpha \in \mathbb{Z}^n} |u_\alpha|^2 \langle \alpha \rangle^{2s} < \infty, \quad (7)$$

where

$$\langle \alpha \rangle := (1 + |\alpha|^2)^{1/2}. \quad (8)$$

The two definitions are clearly equivalent.

Note that being an element of (6) for nonnegative s means that we have some decay of the α -modes. For example for constant α -modes the corresponding function in classical H^s -space

$$\sum_{\alpha \in \mathbb{Z}^n} C \exp\left(\frac{2\pi i \alpha x}{l}\right) \quad (9)$$

is a formal expression of C times the δ distribution and we have

$$\delta \in H^s \text{ if } s < -\frac{n}{2}. \quad (10)$$

Note that we may consider n -tori which are build by identification of opposite sides of the n -cube $[0, l]^n$ not $[-l, l]^n$. This means that we have Fourier representations in the space of even functions, i.e., we may assume that $u_{-\alpha} = u_\alpha$. We make this remark because it is a useful fact for computation, but it is not needed for the proof of our main theorem. Now we may

rephrase the ideas in [7], [8], [10], and [11] in this context. We shall deviate from these schemes as we simplify the control function and define an iterated scheme of global equations corresponding to linear partial integro differential equations approximating the (controlled) incompressible Navier Stokes equation. Formally, the modes $(v_{i\alpha})_{\alpha \in \mathbb{Z}^n}$, $1 \leq i \leq n$, of the velocity function v_i , $1 \leq i \leq n$ of the incompressible Navier-Stokes equation satisfy the infinite ODE-system (derivation below)

$$\begin{aligned} \frac{dv_{i\alpha}}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)} v_{i\gamma} \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (11)$$

where the modes $v_{i\alpha}$ depend on time t and such that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$v_{i\alpha}(0) = h_{i\alpha}. \quad (12)$$

In the infinite system and for fixed α we call the equation in (11) with left side $\frac{dv_{i\alpha}}{dt}$ the α -mode equation. Some simple but important observations are in order here. First note that the damping

$$\sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \quad (13)$$

is not equal to zero unless $|\alpha| = \sum_{i=1}^n |\alpha_i| = 0$. Hence we have damping except for the zero-modes v_{i0} (where the subscript 0 denotes the n -tuple of zeros). Second, note that the zero modes v_{i0} contribute to the α -mode equation for $\alpha \neq 0$ only via the convection terms. This is because for the second term on the right side of (11) only the term with $\gamma = \alpha$, i.e., the summand

$$- \sum_{j=1}^n \frac{2\pi i \alpha_j}{l} v_{j0} v_{i\alpha} \quad (14)$$

contains a zero mode. Third, note that the pressure term in (11) has no zero mode summands since

$$\begin{aligned} 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2} = \\ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha, 0\}} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (15)$$

Furthermore, note that the 0-mode equation consists only of terms corresponding to convection terms, i.e., we have

$$\frac{dv_{i0}}{dt} = - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0\}} \frac{2\pi i \gamma_j}{l} v_{j(-\gamma)} v_{i\gamma}. \quad (16)$$

Furthermore we observe that in (16) we have no zero modes on the right side but only non zero-modes of coupled equations. Furthermore all non-zero modes involve a damping term, because $\nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) > 0$ for $\nu > 0$ and $|\alpha| \neq 0$. Note that damping becomes stronger as the order of the modes $|\alpha| \neq 0$ increases - an important difference to the incompressible Euler equation, where we have no damping indeed. This is also a first hint that it may be useful to define a controlled incompressible Navier-Stokes equation on the torus where a simple control function controls just the zero modes. Next we define schemes for computing the modes. For purposes of global existence it can make sense to start with the multivariate Burgers equation, because we know that we have a unique global regular solution for this equation on the n -torus. This can be derived by the a priori estimates

$$\frac{\partial}{\partial t} \|u(t, \cdot)\|_{H^s} \leq \|u(t, \cdot)\|_{H^{s+1}} \sum_{i,j} \sum_{|\alpha|+|\beta| \leq s} \|D^\alpha u_i D^\beta u_j\|_{L^2} - 2\|\nabla u\|_{H^s}^2, \quad (17)$$

which hold on the n -torus, or by the arguments which we discussed in [9] and [10]. In our context this means that for positive viscosity $\nu > 0$ and smooth data (at least formally) we have a global regular solution for the system

$$\frac{du_{i\alpha}}{dt} = \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) u_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} u_{j(\alpha-\gamma)} u_{i\gamma} \quad (18)$$

where $v_{i\alpha}$ depend on time t and such that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$u_{i\alpha}(0) = h_{i\alpha}. \quad (19)$$

This suggests the following first approach concerning a (formal) scheme for the purpose of global existence. We compute first $v_{i\alpha}^0 = u_{i\alpha}$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ and then iteratively

$$v_{i\alpha}^k := v_{i\alpha}^0 + \sum_{p=1}^k \delta v_{i\alpha}^p, \quad (20)$$

where $\delta v_{i\alpha}^p := v_{i\alpha}^p - v_{i\alpha}^{p-1}$ for all $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ and for $p \geq 1$ we have

$$\begin{aligned} \frac{dv_{i\alpha}^p}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^p v_{i\gamma}^p \\ &\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (21)$$

where $v_{i\alpha}$ depend on time t and such that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$v_{i\alpha}(0) = h_{i\alpha}. \quad (22)$$

Alternatively, we may start with the initial data h_i and their modes $h_{i\alpha}$ as first order coefficients of the first approximating equation. In this case we have for an iteration number $p \geq 0$ the approximation at the p th stage via the linear equation

$$\begin{aligned} \frac{dv_{i\alpha}^p}{dt} = \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{p-1} v_{i\gamma}^p \\ + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (23)$$

where again for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$v_{i\alpha}(0) = h_{i\alpha}. \quad (24)$$

For $p = 0$ we then have $v_{j(\alpha-\gamma)}^{p-1} = v_{j(\alpha-\gamma)}^{-1} := h_{j(\alpha-\gamma)}$. The latter equation in (23) still corresponds to a partial integro-differential equation. However, this scheme has the advantage that we can build an algorithm on it via a Trotter product formula for infinite matrices (because we know the data $h_{i\alpha}$ and for $p \geq 1$ the data $v_{i\alpha}^{p-1}$ from the previous iteration step).

Remark 1.1. The equation in (21) corresponds to a linear integro-differential equation in classical space and differs in this respect from the approximations we considered in [10] and [11] and also in [8] and [7] (although we mentioned this type of global scheme in [7]). The analysis in all these papers was based on local equations because a priori estimates are easier at hand - as is the trick with the adjoint. In dual spaces of Fourier basis representation considered in this paper spatially global equations can be analysed easier. We could also use the corresponding local equations. The corresponding system is

$$\begin{aligned} \frac{dv_{i\alpha}^p}{dt} = \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{p-1} v_{i\gamma}^p \\ + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (25)$$

but there is no real advantage analyzing (25) instead of (21).

Note that the 'global' term in (21) is

$$2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2}, \quad (26)$$

and it looks more like its 'local' companion in this discrete Fourier-based representation than is the case for representations in classical spaces. Formally,

for all $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ and for $p \geq 1$ for $\delta v_{i\alpha}^p = v_{i\alpha}^p - v_{i\alpha}^{p-1}$, $\alpha \in \mathbb{Z}^n$ we have

$$\begin{aligned}
\frac{d\delta v_{i\alpha}^p}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \delta v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{p-1} \delta v_{i\gamma}^p \\
&\quad - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} \delta v_{j(\alpha-\gamma)}^{p-1} v_{i\gamma}^{p-1} \\
&\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2}}{\sum_{i=1}^n 4\pi \alpha_i^2} \\
&\quad - 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-2} v_{k(\alpha-\gamma)}^{p-1}}{\sum_{i=1}^n 4\pi \alpha_i^2}}{\sum_{i=1}^n 4\pi \alpha_i^2}
\end{aligned} \tag{27}$$

where $v_{i\alpha}$ depend on time t and such that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$\delta v_{i\alpha}(0) = 0. \tag{28}$$

It is well-known that the viscosity parameter may be chosen arbitrarily. This fact facilitates the analysis a bit, but for the proof of global existence it is not essential. It is important that the viscosity is strictly positive ($\nu > 0$). We need this for the Trotter product formula below, and without it the arguments breaks down. It is also useful to have a stronger damping for simulations- so let us be more explicit about this. Consider a parameter transformation of the form

$$v_i(t, x) = v_i^*(\tau, y) = r^\lambda v_i^*(r^{\nu'} t, r^\mu x), \quad p(t, x) = p^*(\tau, y) = r^\delta v_i^*(r^{\nu'} t, r^\mu x) \tag{29}$$

for some positive real number $r > 0$ and some positive real parameters λ, μ, ν' . We have (using Einstein notation for spatial variables)

$$\begin{aligned}
\frac{\partial}{\partial t} v_i(t, x) &= r^{\lambda+\nu'} \frac{\partial}{\partial \tau} v_i^*(\tau, y), \quad v_{i,j}(t, x) = r^{\lambda+\mu} v_{i,j}^*(\tau, y), \\
v_{i,j,k}(t, x) &= r^{\lambda+2\mu} \frac{\partial^2}{\partial x_j \partial x_k} v_i^*(\tau, y), \quad p_{,i}(t, x) = r^{\delta+\mu} p_{,i}^*(\tau, y).
\end{aligned} \tag{30}$$

Hence

$$r^{\lambda+\nu'} \frac{\partial}{\partial \tau} v_i^*(\tau, y) = \nu r^{\lambda+2\mu} \Delta v_i^*(\tau, y) - \sum_{j=1}^n r^{2\lambda+\mu} v_j^*(\tau, y) \frac{\partial v_i^*}{\partial x_j}(\tau, y) = r^{\delta+\mu} p_{,i} \tag{31}$$

which (for example) for the parameter constellations

$$\mu, \nu', \lambda, \delta \text{ with } \lambda + \mu - \nu' = 0 \text{ and } \delta + \mu - \lambda - \nu' = 0 \tag{32}$$

becomes

$$\begin{aligned}
& \frac{\partial}{\partial \tau} v_i^*(\tau, y) = \\
& \nu r^{2\mu-\nu'} \Delta v_i^*(\tau, y) - \sum_{j=1}^n r^{\lambda+\mu-\nu'} v_j^*(\tau, y) \frac{\partial v_i^*}{\partial x_j}(\tau, y) - r^{\delta+\mu-\lambda-\nu'} p_{,i} \quad (33) \\
& = \nu r^{2\mu-\nu'} \Delta v_i^*(\tau, y) - \sum_{j=1}^n v_j^*(\tau, y) \frac{\partial v_i^*}{\partial x_j}(\tau, y) - p_{,i}.
\end{aligned}$$

Note that we may choose (for example) $2\mu - \nu' \in \mathbb{R}_+$ (\mathbb{R}_+ being the set of strictly positive real numbers) freely and still satisfy the conditions (32). Note that we can take $\nu' = \nu$. Hence, we are free to assume any viscosity $\nu > 0$. Concerning efficiency of the corresponding algorithm such an observation can be useful, although it is of limited use when it comes to more realistic models of fluids with boundaries and free boundaries. For the numerical and computational analysis it is useful to refine this a bit and consider certain bi-parameter transformation with respect to the viscosity constant $\nu > 0$ and with respect to the diameter l of the n -torus. From an analytic perspective we may say that it is sufficient to prove global existence for specific parameters $\nu > 0$ and $l > 0$. Let us consider coordinate transformations with parameter $l > 0$ first. If \mathbf{v}^1, p^1 is solution on the domain $[0, \infty) \times \mathbb{T}_1^n$, then the function pair $(t, x) \rightarrow \mathbf{v}^l(t, x)$, and $(t, x) \rightarrow p^l(t, x)$ on $[0, \infty) \times \mathbb{T}_l^n$ along with

$$\mathbf{v}^l(t, x) := \frac{1}{l} \mathbf{v}^1\left(\frac{t}{l^2}, \frac{x}{l}\right), \quad (34)$$

and

$$p^l(t, x) := \frac{1}{l^2} p^1\left(\frac{t}{l^2}, \frac{x}{l}\right) \quad (35)$$

is a solution pair on the n -torus of size $l > 0$ with initial data

$$\mathbf{h}^l(x) := \frac{1}{l} \mathbf{h}^1\left(\frac{x}{l}\right) \text{ on } \mathbb{T}_l^n. \quad (36)$$

Hence if we can construct solutions to the Navier-Stokes solution for $h_l \in C^\infty(\mathbb{T}_l^n)$ without further restrictions on the data \mathbf{h}_l , then we can construct global solutions for the problem in \mathbb{T}_1^n with $\mathbf{h} = \mathbf{h}_1 \in C^\infty \in (\mathbb{T}^n)_1$ with factor l as in (36). We only need to produce a fixed number $l > 0$ such that arbitrary data are allowed. Second if v_i^ν , $1 \leq i \leq n$ is a solution to the incompressible Navier Stokes equation with parameter $\nu > 0$ then via the time transformation

$$v_i(t, x) := (r\nu)^{-1} v_i^\nu((r\nu)^{-1}t, x), \quad p(t, x) := (r\nu)^{-2} p^\nu((r\nu)^{-1}t, x) \quad (37)$$

(for all $t \geq 0$) we observe that v_i is solution of the incompressible Navier-Stokes equation with viscosity parameter which may be chosen. Concatenation of the previous transformations shows that we can indeed choose specific

values for $\nu > 0$ and $l > 0$, and this may be useful for designing algorithms. For example, it is useful to observe that in (27) the modulus of the diagonal coefficients

$$\sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \quad (38)$$

becomes large for $|\alpha| \neq 0$ if ν is large or l is small in comparison to the other terms of the iteration in (27), but it is even more interesting that we may choose ν large and $l > 0$ large on a scale such that the convection terms are small in comparison to the diagonal damping terms in (38). As we observed in other articles for the time-local solution of the Navier-Stokes equation we may set up a scheme involving simple scalar linear parabolic equations of the form (39) (cf. [7, 8, 9, 10, 11]). We maintained that in classical space a global controlled scheme for an equivalent equation can be obtained if in a time-local scheme the approximate functions in a functional series (evaluated at arbitrary time t) inherit the property of being in $C^k \cap H^k$ for $k \geq 2$ together with polynomial decay of order k . This is remarkable since you will not expect this from standard local a priori estimates of the Gaussian. For the equivalent controlled scheme in [11] we only needed to prove this for the higher order correction terms of the time-local scheme. There it is true for some order of decay because the representations involve convolution integrals with products of approximative value functions where each factor is inductively of polynomial decay. The growth of the Leray projection term for the scheme in [11] is linearly bounded on a certain time scale. In [7, 8] we proposed more complicated equivalent controlled schemes with a bounded regular control function which controls the growth of the controlled solution function such that it stays bounded for all time. Actually, the simple schemes may be reconsidered in terms of the Trotter product formula representations of solutions to scalar parabolic equations of the form

$$\frac{\partial u}{\partial t} - \nu \Delta u + Wu = g, \quad (39)$$

with initial data $u(0, x) = f \in \cap_{s \in \mathbb{R}^n} H^s$ and some dynamically generated source terms g , and where $W = \sum_{i=1}^n w_i(x) \frac{\partial}{\partial x_i}$ is a vector field with bounded and uniformly Lipschitz continuous coefficients. The w_i then are replaced by approximative value function components at each iteration step of course. Local application of the Trotter product formula leads to representations of the form

$$\exp(t(\nu\Delta + W))f = \lim_{k \uparrow \infty} \left(\exp\left(\frac{t}{k}W\right) \exp\left(\frac{t}{k}\nu\Delta\right) \right)^k f. \quad (40)$$

Then in a second step having obtained time-local representations of solutions to the incompressible Navier-Stokes equation one can use this reduction from a system to a scalar level and apply the Trotter product formula on a global

time scale. From this point of view the local contraction results considered in [11] or in [9] (with the use of adjoint equations) are essential. In dual spaces the use of a Trotter product formula seems to be not only useful but mandatory in two respects. Consider the infinite matrix

$$D := (d_{\alpha\beta})_{\alpha,\beta \in \mathbb{Z}^n} := \left(\delta_{\alpha\beta} \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \right)_{\alpha,\beta \in \mathbb{Z}^n} \quad (41)$$

with the infinite Kronecker delta function $\delta_{\alpha\beta}$, i.e.,

$$\delta_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \quad (42)$$

If we measure regularity with respect to the the degree of decay of entries as the order of the modes increases, then this matrix lives in a space of rather weak regularity. Worse then this, iterations of the matrix, i.e., matrices of the form

$$D^2 := DD =: (d_{\alpha\beta}^{(2)})_{\alpha,\beta \in \mathbb{Z}^n} := \left(\sum_{\gamma \in \mathbb{Z}^n} d_{\alpha\gamma} d_{\gamma\beta} \right)_{\alpha,\beta \in \mathbb{Z}^n} \quad (43)$$

$$D^m := DD^{m-1} =: (d_{\alpha\beta}^{(m)})_{\alpha,\beta \in \mathbb{Z}^n} := \left(\sum_{\gamma \in \mathbb{Z}^n} d_{\alpha\gamma} d_{\gamma\beta}^{(m-1)} \right)_{\alpha,\beta \in \mathbb{Z}^n}$$

live in matrix spaces of lower and lower regularity as m increases (if we measure regularity in relation to decay with respect to the order of modes). However, since the matrix D has a minus sign we may make sense of the matrix

$$\exp(D) = (\delta_{\alpha\beta} \exp(d_{\alpha\alpha}))_{\alpha,\beta \in \mathbb{Z}^n} = \left(\delta_{\alpha\beta} \exp \left(\sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \right) \right)_{\alpha,\beta \in \mathbb{Z}^n}, \quad (44)$$

and this matrix makes perfect sense in terms of the type of regularity mentioned, i.e. the type of regularity expressed by dual Sobolev spaces which measure the order of polynomial decay of the modes. Since we consider infinite ODEs involving a Leray projection term we shall have correlations between the components $1 \leq i \leq n$ even at each approximation step. Hence our matrices shall involve big diagonal matrices of the form

$$(\delta_{ij} \exp(D))_{1 \leq i,j \leq n} \quad (45)$$

where δ_{ij} denotes the usual Kronecker δ for $1 \leq i,j \leq n$ and $\exp(D)$ is as in (44) above. However, the idea that such a dissipative matrix lives in a regular (\equiv 'polynomially decaying modes as the order of modes increases') infinite matrix space is the same.

Now consider the first step in our scheme for the vectors $(v_{i\alpha}^p)_{\alpha \in \mathbb{Z}^n}$. For $p = 0$ the modes $(v_{i\alpha}^0)_{\alpha \in \mathbb{Z}^n}$ are equal to the modes of the controlled scheme we describe below and the corresponding equation can be written in the form

$$\begin{aligned} \frac{dv_{i\alpha}^0}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^0 - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} h_{j(\alpha-\gamma)} v_{i\gamma}^0 \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^0 h_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (46)$$

If we consider the n -tuple $\mathbf{v}^F = (\mathbf{v}_1^F, \dots, \mathbf{v}_n^F)^T$ as an infinite vector with $\mathbf{v}_i^F := (v_{i\alpha})_{\alpha \in \mathbb{Z}^n}$, then with the usual identifications the equation (46) is equivalent to an infinite linear ODE

$$\frac{d\mathbf{v}^{0,F}}{dt} = A_0 \mathbf{v}^{0,F}, \quad (47)$$

where the matrix A_0 is defined implicitly given by (46) and will be given explicitly in our more detailed description below. Together with the initial data

$$\mathbf{v}^{0,F}(0) = \mathbf{h}^F(0) \quad (48)$$

this is an equivalent formulation of the equation for the first step of our scheme. We shall define a dissipative diagonal matrix D_0 and a matrix B_0 related to the convection and Leray projection terms such that $A_0 = D_0 + B_0$, and prove a Trotter product formula which allows us to make sense of the formal solution

$$\begin{aligned} \mathbf{v}^{0,F}(t) &= \exp(A_0 t) \mathbf{h}^F \\ &:= \lim_{l \uparrow \infty} \lim_{k \uparrow \infty} \left(\exp \left(P_{M^l} (D_0) \frac{t}{k} \right) \exp \left(P_{M^l} B_0 \frac{t}{k} \right) \right)^k \mathbf{h}^F. \end{aligned} \quad (49)$$

Here P_{M^l} denotes a projection to the finite modes of order less or equal to $l > 0$. Here and in the following we may assume that we have some order of the multiindices and that this order is preserved by the projection operators. Note that at this first stage the modes $h_{i\alpha}$ are not time-dependent. Hence A_0 is defined as a matrix which is independent of time, and we have no need of a Dyson formalism at this stage. For the higher stages of approximation a control function comes in and we have to deal with time dependence of the related infinite matrices A_p which define the infinite ODEs at iteration step $p \geq 0$ for the modes $v_{i\alpha}^p$ and $v_{i\alpha}^{r,p}$ in the presence of a control function r . The formula (40) makes clear why strong contraction estimates of the time-local expressions

$$\exp \left(\frac{t}{k} W \right) \exp \left(\frac{t}{k} \nu \Delta \right) f \quad (50)$$

are important. As we said, we pointed out this in [10] and [11] from a different point of view. In dual spaces we may approximate such expressions via matrix equations of finite modes (projections of expressions of the form (50) to approximating equations of finite modes). For the finite mode approximation of the first approximation sequence $v_{i\alpha}^0$, $\alpha \in \mathbb{Z}^n$ first order coefficients are time independent and we may use the Baker-Campbell-Hausdorff formula for finite matrices A and B of the form

$$\exp(A) \exp(B) = \exp(C), \quad (51)$$

where

$$\begin{aligned} C = & A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, [A, B]] + \frac{1}{12} [[A, B], B] \\ & + \text{higher order terms.} \end{aligned} \quad (52)$$

Here $[\cdot, \cdot]$ denote Lie brackets and the expression 'higher order terms' refers to all terms of multiple commutators of A and B . We shall see that for dissipative operators as in the case of the incompressible Navier-Stokes equation and its linear approximations we can prove extensions of the formula in (51) to infinite matrices, where we restrict our investigation to the special cases which fit for the analysis of some infinite linear ODEs approximating the incompressible Navier-Stokes equation written in dual space. Note that for higher order approximations $v_{i\alpha}^p$, $\alpha \in \mathbb{Z}^n$, $1 \leq i \leq n$ with $p \geq 1$ we have time dependence of the coefficients and this means that we have to apply a time order operator as in Dyson's formalism in order to solve the related linear approximating equations formally. Note that (52) is closely connected to the Hörmander condition, and this was one of the indicators which leads to the expectation in [8] that global smooth existence is true if the Hörmander condition is satisfied. This will be considered elsewhere.

In this paper we shall see that we can simplify the schemes formulated in classical spaces in our formulation on dual spaces in some respects. The first simplification is that we may define an iteration scheme on a global time scale. The second simplification is that the estimates in dual spaces become estimates of discrete infinite sums which can be done on a very elementary level. On the other hand we shall need a control function which ensures that the scheme for the controlled equation is a scheme for non-zero modes (ensuring that the damping factors associated with strictly positive viscosity or dissipative effects are active for all modes of the controlled scheme). We shall choose specific ν and l and define a controlled scheme for a controlled incompressible Navier-Stokes equation such that the diagonal matrix elements corresponding to the Laplacian terms become dominant and such that there exists a global iteration in an appropriate function space. However the control function is a scalar univariate function and is much simpler than the control functions considered in [7] and [8]. Indeed, the control func-

tion in the present paper will only control the zero modes v_{i0} of the value function modes.

Next we define the controlled scheme. We may start with one of the two possibilities mentioned above. We may start with the solution scheme for the multivariate Burgers equation, which is given in dual representation by the infinite ODE

$$\frac{du_{i\alpha}}{dt} = \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) u_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} u_{j(\alpha-\gamma)} u_{i\gamma}, \quad (53)$$

where the modes $u_{i\alpha}$ depend on time t and such that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$u_{i\alpha}(0) = h_{i\alpha}. \quad (54)$$

This is a slight variation of the linearized scheme mentioned above. We may define $v_{i\alpha}^0 = u_{i\alpha}$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ and then iteratively

$$v_{i\alpha}^k := v_{i\alpha}^0 + \sum_{p=1}^k \delta v_{i\alpha}^p, \quad (55)$$

where for $p \geq 1$ we define $\delta v_{i\alpha}^p := v_{i\alpha}^p - v_{i\alpha}^{(p-1)}$ for all $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ such that

$$\begin{aligned} \frac{d\delta v_{i\alpha}^p}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \delta v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{p-1} \delta v_{i\gamma}^p \\ &\quad - \sum_{j=1}^n \frac{2\pi i \alpha_j}{l} v_{j0}^{p-1} \delta v_{i\alpha}^p - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} \delta v_{j(\alpha-\gamma)}^{p-1} v_{i\gamma}^{p-1} \\ &\quad - \sum_{j=1}^n \frac{2\pi i \alpha_j}{l} \delta v_{j0}^{p-1} v_{i\alpha}^{p-1} \\ &\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{p-1} \delta v_{k(\alpha-\gamma)}^p}{\sum_{i=1}^n 4\pi \alpha_i^2} \\ &\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) \delta v_{j\gamma}^{p-1} v_{k(\alpha-\gamma)}^{p-1}}{\sum_{i=1}^n 4\pi \alpha_i^2} \end{aligned} \quad (56)$$

(where for $p = 0$ and $p = -1$ we define $\delta v_{i\alpha}^p = 0$). Note that we extracted the zero modes from the sums in (56). The reason is that we want to extend this scheme to a controlled scheme which is equivalent but has no zero modes. The present local scheme suggests this because the Leray projection terms do not contain zero modes. Note that for all $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we have

$$\delta v_{i\alpha}^p(0) = 0. \quad (57)$$

Global smooth existence means that the $v_{i\alpha}^p$ are defined for all time $\mathbb{R}_+ = \{t \in \mathbb{R} | t \geq 0\}$, and such that polynomial decay of the modes is preserved

throughout time. Next we introduce the idea of a simplified control function was introduced in [7] and [8], and [11] in a simplified but still complicated form. Here we introduce a control function for the zero modes. We define

$$v_{i\alpha}^{r,p}(t) = v_{i\alpha}^p(t) + r_0^p(t), \quad (58)$$

where

$$r_0^p : [0, \infty) \rightarrow \mathbb{R} \quad (59)$$

is defined by

$$r_0^p(t) = -v_{i0}^p(t). \quad (60)$$

We have to show then that $r_0 : [0, \infty) \rightarrow \mathbb{R}$ is well-defined (especially bounded), i.e., we have to show that there is a limit

$$r_0(t) := \lim_{p \uparrow \infty} r_0^p(t) = r_0^0(t) + \sum_{p=1}^{\infty} \delta r_0^p(t) \quad (61)$$

along with $\delta r_0^p(t) = r_0^p(t) - r_0^{p-1}(t)$. This leads to the following scheme. We start with the solution for the multivariate Burgers equation but annihilate the zero modes, i.e. we define

$$v_{i\alpha}^{r,0} = v_{i\alpha}^0 = u_{i\alpha} \quad (62)$$

for $\alpha \neq 0$ and

$$v_{i0}^{r,0} = v_{i0}^0 + r_0^0 = 0, \quad (63)$$

where for all $t \geq 0$

$$r_0^0(t) = -u_{i0}(t). \quad (64)$$

For $p \geq 1$ and for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$ we define

$$v_{i\alpha}^{r,k} := v_{i\alpha}^{r,0} + \sum_{p=1}^k \delta v_{i\alpha}^{r,p}, \quad (65)$$

where $\delta v_{i\alpha}^{r,p} := \delta v_{i\alpha}^p + r_{i\alpha}^p$ for all $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$, and

$$\begin{aligned} \frac{d\delta v_{i\alpha}^{r,p}}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \delta v_{i\alpha}^{r,p} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{r,p-1} \delta v_{i\gamma}^{r,p} \\ &\quad - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} \delta v_{j(\alpha-\gamma)}^{r,p-1} v_{i\gamma}^{r,p-1} \\ &\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{r,p-1} \delta v_{k(\alpha-\gamma)}^{r,p-1}}{\sum_{i=1}^n 4\pi \alpha_i^2} \\ &\quad + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) \delta v_{j\gamma}^{r,p-1} v_{k(\alpha-\gamma)}^{r,p-2}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (66)$$

Furthermore, for all $1 \leq i \leq n$ and $\alpha = 0$ we shall ensure that

$$\delta v_{i\alpha}^{r,p}(0) = 0. \quad (67)$$

Note that in the equation (66) the terms

$$-\sum_{j=1}^n \frac{2\pi i \alpha_j}{l} v_{j0}^{p-1} \delta v_{i\alpha}^p - \sum_{j=1}^n \frac{2\pi i \alpha_j}{l} \delta v_{j0}^{p-1} v_{i\alpha}^{p-1} \quad (68)$$

on the right side of (56) are cancelled. This is because we define the control function r_0 such that the zeromodes become zero. This is done as follows. First we note that for $p = 0$ and $p = -1$ we may define $\delta v_{i\alpha}^{r,p} = 0$ for all $\alpha \in \mathbb{Z}^n$. For $\alpha = 0$ we define first the increment $\delta v_{i0}^{*,r,p}$ for $p \geq 1$ via the equation

$$\begin{aligned} \frac{d\delta v_{i0}^{*,r,p}}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) \delta v_{i0}^{*,r,p} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(-\gamma)}^{*,r,p-1} \delta v_{i\gamma}^{*,r,p} \\ & - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} \delta v_{j(-\gamma)}^{*,r,p-1} v_{i\gamma}^{*,r,p-1}. \end{aligned} \quad (69)$$

Then we define

$$\delta r_0^p(t) = -\delta v_{i0}^{*,r,p}, \quad (70)$$

closing the recursion. Note that this ensures

$$\delta v_{i0}^{r,p} = 0 \quad (71)$$

at all stages $p \geq 1$ for the zero modes (provided that we can ensure that r_0^p is globally well-defined). The introduction of the control function for the zero modes makes the system 'autonomous' for the modes $\alpha \in \mathbb{Z}^n \setminus \{0\}$. This means that all modes have damping factors (involving $\nu > 0$), i.e., we have the stabilizing diagonal factor terms

$$\delta_{\alpha\beta} \nu \left(-\sum_{j=1}^n \frac{4\pi\alpha_j^2}{l^2} \right), \quad (72)$$

which contribute for

$$\sum_{j=1}^n \alpha_j^2 \neq 0, \text{ or } \alpha_i \neq 0 \ \forall \ 1 \leq i \leq n. \quad (73)$$

We observed that the parameter $\nu > 0$ may be large without loss of generality. This alone may indicate that the matter of global smooth existence should be detached from the subject of turbulence as simulation indicate turbulent phenomena for high Reynold numbers. Solutions of each approximation step of the scheme can be represented in terms of fundamental

solutions of scalar equations, where each iteration step of higher order involves fundamental solutions of linear scalar parabolic equations. This is an idea which we considered earlier in [9], [10], [8], and [7]. In this paper we consider 'fundamental solutions' in a dual space of Fourier modes. They may be called 'generalized θ -functions' but in general they live only in distributional space of negative Sobolev norm. However, we shall see that for certain initial data we can make sense of the related analytic vectors, and show that there spatial convergence radius is global (holds for the full size of the n -torus).

In this paper we observe that for all data $\mathbf{h} = (h_1, \dots, h_n)$ along with $h_i \in C^\infty$ there exist a number $\nu > 0$ such that the global extended iteration scheme converges globally to a solution of an infinite ODE system equivalent to a controlled Navier-Stokes equation in a strong norm, i.e. for all $t \geq 0$ the sequences $v_{i\alpha}$, $\alpha \in \mathbb{Z}^n$ converge in the dual Sobolev space $h^s(\mathbb{Z}^n)$ for $s \geq 2$. Moreover, the control function is a globally well-defined univariate bounded differentiable function such that controlled incompressible Navier-Stokes equation is equivalent to the usual incompressible Navier-Stokes equation system. More precisely, we have

Theorem 1.2. *There are numbers $l > 0$ and $\nu > 0$ such that for all $1 \leq p \leq n$ and data $\mathbf{h} \in [C^\infty(\mathbb{T}_l^n)]^n$ for all $s \in \mathbb{R}$ we have*

$$v_{i\alpha}^r(t) = \lim_{m \uparrow \infty} \mathbf{v}_{i\alpha}^{r,m}(t) \in h_l^s(\mathbb{Z}^n). \quad (74)$$

for the scheme described in the introduction. Moreover, $t \rightarrow v_{i\alpha}^r(t)$ satisfies the infinite ODE system in (56). This implies that $v_{i\alpha}(t) = v_{i\alpha}^r(t) - \delta_{0\alpha} r_i(t)$ satisfies the infinite ODE-system corresponding to the incompressible Navier-Stokes equation, where $\delta_{0\alpha} = 0$ if $\alpha \neq 0$ and $\delta_{\alpha 0} = 1$ if $\alpha = 0$, i.e., the function

$$v_j := \sum_{\alpha \in \mathbb{Z}^n} v_{j\alpha} \exp(2\pi i \alpha x), \quad 1 \leq j \leq n, \quad (75)$$

is a global classical solution of the Navier stokes equation system (76) below. Note that $v_{j\alpha}$, $\alpha \in \mathbb{Z}^n$ are functions of time as is v_j in (75).

Here we say that $\mathbf{v} = (v_1, \dots, v_n)^T$ is a global classical solution of the incompressible Navier-Stokes equation system

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p & t \geq 0, \\ \nabla \cdot \mathbf{v} = 0, & t \geq 0, \\ \mathbf{v}(0, x) = \mathbf{h}(x), \end{cases}, \quad (76)$$

on some domain Ω (which is the n -torus \mathbb{T}^n in this paper) if \mathbf{v} solves the equivalent Navier-Stokes equation system in its Leray projection form, where

p is eliminated by the Poisson equation

$$-\Delta p = \sum_{j,k=1}^n v_{j,k} v_{k,j}. \quad (77)$$

Note that $v_{j,k}$ denotes the first partial derivative of v_j with respect to the component x_k etc. (Einstein notation). This means that we construct a solution to an equation of the form

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} - \nu \Delta \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \sum_{j,k=1}^n \int \nabla K(x-y) v_{j,k} v_{k,j}(t,y) dy \\ \mathbf{v}(0, x) = \mathbf{h}(x), \end{cases} \quad (78)$$

along with (77). For a fixed function \mathbf{v} which solves (78) with (77) the Cauchy problem for divergence

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{div} \mathbf{v} - \nu \Delta \operatorname{div} \mathbf{v} + \sum_{j=1}^n v_j \operatorname{div} \mathbf{v} + \sum_{j,k=1}^n v_{j,k} v_{k,j} &= -\Delta p \\ \operatorname{div} v(0, \cdot) &= 0 \end{aligned} \quad (79)$$

simplifies to

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{div} \mathbf{v} - \nu \Delta \operatorname{div} \mathbf{v} + \sum_{j=1}^n v_j \operatorname{div} \mathbf{v} &= 0 \\ \operatorname{div} \mathbf{v}(0, \cdot) &= 0, \end{aligned} \quad (80)$$

yielding $\operatorname{div} \mathbf{v} = 0$ as the unique solution of (80). For this reason it suffices to solve the Navier-Stokes equation in its Leray projection form (as is well-known). In the next Section we look at the structure of the proof of the main theorems. Then in Section 3 we prove theorem 1.2.

Corollary 1.3. *The statements of (1.3) hold for all numbers $l > 0$ and $\nu > 0$. Furthermore, the solution is unique in the function space*

$$F_{ns} = \left\{ \mathbf{g} : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n \mid g_i \in C^1([0, \infty), H^s(\mathbb{R}^n)) \text{ \& } g_i(0, \cdot) = h_i \right\}, \quad (81)$$

where $\mathbf{g} = (g_1, \dots, g_n)^T$ and $h_i \in H^s$ for $s > n \geq 3$.

Remark 1.4. For the assumptions of (1.3) the methods of this paper lead directly to contraction results. Concerning uniqueness the regularity assumptions of the initial data may be weakened, of course.

2 Structure and ideas of the proof of theorem 1.2 and corollary 1.3

In the proof of theorem 1.2 in section 3 below we first recall that the incompressible Navier-Stokes equation is formally equivalent to a system of n

infinite ODE-systems

$$\begin{aligned} \frac{dv_{i\alpha}}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)} v_{i\gamma} \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2}}, \end{aligned} \quad (82)$$

where $\alpha \in \mathbb{Z}^n$, and where the initial data $\mathbf{v}_i^F(0) = (v_{i\alpha}(0))_{\alpha \in \mathbb{Z}^n}$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ are given by n infinite vectors

$$\mathbf{v}_i^F(0) = \mathbf{h}_i^F = (h_{i\alpha})^T. \quad (83)$$

Furthermore, the entries $h_{i\alpha}$ of the latter infinite vector denote the Fourier modes of the functions h_i along with $\mathbf{h} \in [C^\infty(\mathbb{T}_l^n)]^n$. Smoothness of \mathbf{h} translates to polynomial decay of the modes $h_{i\alpha}$. Here we may say that the modes $h_{i\alpha}$ have polynomial decay of order $m > 0$ if

$$|\alpha|^m h_{i\alpha} \downarrow 0 \text{ as } |\alpha| = \sum_{i=1}^n |\alpha_i| \uparrow \infty \quad (84)$$

for a fixed positive integer m , and the modes $h_{i\alpha}$ have polynomial decay if they have polynomial of any order $m > 0$. We may use the terms 'modes of solution have polynomial decay' and 'solution is smooth' interchangeably in our context. We may rewrite the equation (82) in the form

$$\frac{d\mathbf{v}^F}{dt} = A^{NS}(\mathbf{v}) \mathbf{v}^F, \quad (85)$$

where $\mathbf{v}^F = (\mathbf{v}_1^F, \dots, \mathbf{v}_n^F)^T$. Furthermore $A^{NS}(\mathbf{v})$ is a $n\mathbb{Z}^n \times n\mathbb{Z}^n$ -matrix

$$A^{NS}(\mathbf{v}) = (A_{ij}^{NS}(\mathbf{v}))_{1 \leq i, j \leq n} \quad (86)$$

where for $1 \leq i, j \leq n$ the entry $A_{ij}^{NS}(\mathbf{v})$ is a $\mathbb{Z}^n \times \mathbb{Z}^n$ -matrix. We define

$$A^{NS}(\mathbf{v}) \mathbf{v}^F = \left(\sum_{j=1}^n A_{1j}^{NS}(\mathbf{v}) \mathbf{v}_j^F, \dots, \sum_{j=1}^n A_{nj}^{NS}(\mathbf{v}) \mathbf{v}_j^F \right)^T, \quad (87)$$

where for all $1 \leq i \leq n$

$$\sum_{j=1}^n A_{ij}^{NS}(\mathbf{v}) \mathbf{v}_j^F = \left(\left(\sum_{j=1}^n \sum_{\beta \in \mathbb{Z}^n} A_{i\alpha j \beta}^{NS}(\mathbf{v}) v_{j\beta} \right)_{\alpha \in \mathbb{Z}^n} \right)_{1 \leq i \leq n}^T. \quad (88)$$

The entries $A_{i\alpha j \beta}^{NS}(\mathbf{v})$ of $A^{NS}(\mathbf{v})$ are determined by the equation in (82) of course. On the diagonal, i.e., for $i = j$ we have the entries

$$\begin{aligned} \delta_{ij} A_{i\alpha j \beta}^{NS}(\mathbf{v}) = & \delta_{ij} \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) - \delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{l} v_{j(\alpha-\beta)} \\ & + \delta_{ij} 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (89)$$

where $\delta_{ij} = 1$ iff $i = j$ and zero else denotes the Kronecker δ -function, and off-diagonal we have for $i \neq j$ the entries

$$(1 - \delta_{ij})A_{i\alpha j\beta}^{NS}(\mathbf{v}) = 2\pi i\alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi\beta_j(\alpha_k - \beta_k)v_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi\alpha_i^2}. \quad (90)$$

Here, we remark again that the i in the context $2\pi i$ denotes the complex number $i = \sqrt{-1}$, while the other i th are integer indices. The next step is to define a controlled system according to the ideas described in the introduction. Recall the main idea: the control function cancels the zero modes in an iterative scheme. If this is possible and a regular limit exists, then it cancels the zero modes in the limit as well. Consider the representation in (82). The zero modes do not appear in the dissipative term related to the Laplacian, and we have observed that we may represent the Leray projection term without zero mode terms as well. Hence, for the zero modes v_{i0} in the equation in (82) we have for $\alpha = 0$

$$\frac{dv_{i0}}{dt} = -\sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i\gamma_j}{t} v_{j(-\gamma)} v_{i\gamma}. \quad (91)$$

Assuming that the function $t \rightarrow v_{i0}(t)$ is bounded it is natural to define

$$r_0^i(t) := -v_{i0}(t). \quad (92)$$

Formally, this leads to the controlled equation for $v_{i\alpha}^r$, $\alpha \in \mathbb{Z}^n \setminus \{0\} = \mathbb{Z}^{n,0}$.

$$\begin{aligned} \frac{dv_{i\alpha}^r}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{t^2} \right) v_{i\alpha}^r - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} \frac{2\pi i\gamma_j}{t} v_{j(\alpha-\gamma)}^r v_{i\gamma}^r \\ & + 2\pi i\alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} 4\pi\gamma_j(\alpha_k - \gamma_k) v_{j\gamma}^r v_{k(\alpha-\gamma)}^r}{\sum_{i=1}^n 4\pi\alpha_i^2}, \end{aligned} \quad (93)$$

where $\alpha \in \mathbb{Z}^n$, and where the initial data $\mathbf{v}_i^{r,F}(0) = (v_{i\alpha}^r(0))_{\alpha \in \mathbb{Z}^n \setminus \{0\}}$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$ are given by n infinite vectors

$$\mathbf{v}_i^{r,m,F}(0) = \mathbf{h}_i^{r,F} = (h_{i\alpha})_{\alpha \in \mathbb{Z}^n \setminus \{0\}}^T. \quad (94)$$

Note that

$$v_{i0} + r_0 = 0, \quad (95)$$

such that we may cancel the zero modes. The justification for this is obtained for each iteration step m below. We may rewrite the equation (93) in the form

$$\frac{d\mathbf{v}^{r,F}}{dt} = A^{r,NS}(\mathbf{v}) \mathbf{v}^{r,F}, \quad (96)$$

where $\mathbf{v}^{r,F} = \left(\mathbf{v}_1^{r,F}, \dots, \mathbf{v}_n^{r,F} \right)^T$. Furthermore $A^{r,NS}(\mathbf{v})$ is a $n\mathbb{Z}^n \times n\mathbb{Z}^n$ -matrix

$$A^{r,NS}(\mathbf{v}) = \left(A_{ij}^{r,NS}(\mathbf{v}) \right)_{1 \leq i,j \leq n} \quad (97)$$

where for $1 \leq i, j \leq n$ the entry $A_{ij}^{r,NS}(\mathbf{v})$ is a $\mathbb{Z}^n \times \mathbb{Z}^n$ -matrix. We define

$$A^{r,NS}(\mathbf{v}) \mathbf{v}^F = \left(\sum_{j=1}^n A_{1j}^{r,NS}(\mathbf{v}) \mathbf{v}_j^{r,F}, \dots, \sum_{j=1}^n A_{nj}^{r,NS}(\mathbf{v}) \mathbf{v}_j^{r,F} \right)^T, \quad (98)$$

where for all $1 \leq i \leq n$

$$\sum_{j=1}^n A_{ij}^{r,NS}(\mathbf{v}) \mathbf{v}_j^{r,F} = \left(\left(\sum_{j=1}^n \sum_{\beta \in \mathbb{Z}^n} A_{i\alpha j\beta}^{r,NS}(\mathbf{v}) v_{j\beta}^r \right)_{\alpha \in \mathbb{Z}^n} \right)_{1 \leq i \leq n}^T. \quad (99)$$

The entries $A_{i\alpha j\beta}^{r,NS}(\mathbf{v})$ of $A^{r,NS}(\mathbf{v})$ are determined by the equation in (93) of course. On the diagonal, i.e., for $i = j$ we have the entries for $\alpha, \beta \neq 0$

$$\begin{aligned} \delta_{ij} A_{i\alpha j\beta}^{r,NS}(\mathbf{v}) &= \delta_{ij} \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) - \delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{l} v_{j(\alpha-\beta)}^r \\ &+ \delta_{ij} 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^r}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (100)$$

where for $\alpha = \beta$ the terms of the form $v_{k(\alpha-\beta)}^r$ are zero (such that we do not need to exclude these terms explicitly). Furthermore, off-diagonal we have for $i \neq j$ the entries

$$(1 - \delta_{ij}) A_{i\alpha j\beta}^{r,NS}(\mathbf{v}) = 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^r}{\sum_{i=1}^n 4\pi \alpha_i^2}. \quad (101)$$

The idea for a global scheme is to determine $\mathbf{v}^{r,F} = \lim_{m \uparrow \infty} \mathbf{v}^{r,m,F}$ for a simple control function and a certain iteration

$$\frac{d\mathbf{v}^{r,m,F}}{dt} = A^{NS}(\mathbf{v}^{r,m-1}) \mathbf{v}^{r,m,F}, \quad (102)$$

starting with $\mathbf{v}^{r,0} := \mathbf{h}$ or with the information of a global solution of the multivariate Burgers equation. The coefficients in (102) will be time-dependent in general, which means that we shall have to develop a kind of Dyson formalism for the equation. We shall define Trotter product formulae for certain linear infinite equations with time independent coefficients, and then we shall define a time discretization in order to apply these Trotter product formula. The time-dependent linear approximations are then solved by limits.

In order to define a global solution scheme for this system of coupled infinite nonlinear ODEs we may start with the solution of the (viscous) multivariate Burgers equation (where we know that a unique global smooth solution exists), i.e., the solution of

$$\frac{dv_{i\alpha}^B}{dt} = \nu \sum_{j=1}^n \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^B - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^B v_{i\gamma}^B, \quad (103)$$

where $\alpha \in \mathbb{Z}^n$, and where the initial data $\mathbf{v}_i^{BF}(0) = (v_{i\alpha}^B(0))_{\alpha \in \mathbb{Z}^n}$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n$ are given by n infinite vectors

$$\mathbf{v}_i^{BF}(0) = \mathbf{h}_i^F = (h_{i\alpha})^T. \quad (104)$$

We know that the solution $(v_{i\alpha}^B)_{\alpha \in \mathbb{Z}^n, 1 \leq i \leq n}$ is in $h^s(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$ for all time $t \geq 0$. However, without assuming knowledge about the multivariate Burgers equation we may also start with

$$\begin{aligned} \frac{dv_{i\alpha}^0}{dt} = & \nu \sum_{j=1}^n \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^0 - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} h_{j(\alpha-\gamma)} v_{i\gamma}^1 \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^0 h_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (105)$$

where $\alpha \in \mathbb{Z}^n$, and where the initial data $\mathbf{v}^{0,F}(0) = (\mathbf{v}_1^{0,F}(0), \dots, \mathbf{v}_n^{0,F}(0))$ are given by $\mathbf{v}_i^{0,F}(0) = (v_{i\alpha}^0(0))_{\alpha \in \mathbb{Z}^n} = (h_{i\alpha})_{\alpha \in \mathbb{Z}^n}$ for $1 \leq i \leq n$. In the equation (105) and for $\nu > 0$ the zero modes ($|\alpha| = 0$) are the only modes where the damping (viscous) term

$$\nu \sum_{j=1}^n \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^0 \quad (106)$$

cancels. The same holds for the equation in (82) of course. Therefore we introduce the first approximation of a control function $r_{i0}^0 : [0, \infty) \rightarrow \mathbb{R}$ for each $1 \leq i \leq n$ such the scheme for $\mathbf{v}_i^{r,0,F} = \mathbf{v}_i^{0,F} + \mathbf{e}_0 r_{i0}^0$, $1 \leq i \leq n$ becomes a system of nonzero modes. At each stage we have to prove that this is possible, of course, i.e., that the solution of the linear approximative problem exists. Here, $\mathbf{v}_i^{0,F} + \mathbf{e}_0 r_{i0}^0 = (v_{i\alpha}^0)_{\alpha \in \mathbb{Z}^n} = (v_{i\alpha}^{r,0})_{\alpha \in \mathbb{Z}^n} + r_{i0}^0$ is a vector with

$$v_{i\alpha}^{r,0} := \begin{cases} v_{i\alpha}^0 & \text{if } \alpha \neq 0, \\ v_{i0}^0 + r_{i0}^0 & \text{else.} \end{cases} \quad (107)$$

The control function is constructed in such a way that it cancels the zero modes of the original system at each stage of the construction. Hence it is constructed as a series $(r_0^m)_m$ where the convergence as $m \uparrow \infty$ follows from properties of the controlled approximative solution functions $\mathbf{v}^{r,m,F}$. Note that it has to be shown that the control function is a) finite at each stage, and b) that it is finite in the limit of all stages. Note that the Leray projection term does not contribute to the zero modes in (105), and this may already indicates this finiteness. The next step is to analyze the scheme formally defined in the introduction starting from (58) to (70). The effect of this formal scheme is that it is autonomous with respect to the nonzero modes, i.e., only the modes with $|\alpha| \neq 0$ are dynamically active. That the

suppression of the zero modes via a control function is possible, i.e., that the controlled system is well-defined, is shown within the proof. Then we get into the heart of the proof. The iteration describes a set of analytic vectors $(\mathbf{v}^{r,m,F}(t))_{m \in \mathbb{N}} = (\mathbf{v}_i^{r,m,F}(t))_{1 \leq i \leq n, m \in \mathbb{N}}$ with

$$\mathbf{v}^{r,m,F}(t) := T \exp(A_m^r t) \mathbf{h}_i^F, \quad (108)$$

where T is a time order operator (as in Dyson's formalism), and the restrictive dual function spaces together with the dissipative features of the operator A_m^r will make sure that at each time step m the approximation (108) (corresponding to a linear equation) really makes sense. Concerning dissipation features note that we may choose $\nu > 0$ arbitrarily as is well-known and as we pointed out in the introduction. Note that we have a time-order operator T for all stages $m \geq 1$ since the infinite matrices A_m^r depend on time t for $m \geq 1$. In order to make sense of matrix multiplication of infinite unbounded matrices we consider the matrices involved in the computation of the first approximation $\mathbf{v}_i^{r,0,F}(t) = P_0 \mathbf{v}_i^{0,F}(t)$ (where P_0 is the related projection operator which eliminates the zero mode) as an instructive example. We may rewrite (105) in the form

$$\begin{aligned} \frac{dv_{i\alpha}^0}{dt} &= \nu \sum_{j=1}^n \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^0 - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} h_{j(\alpha-\gamma)} v_{i\gamma}^0 \\ &+ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_i (\alpha_k - \gamma_k) h_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2} v_{i\gamma}^0}{\sum_{i=1}^n 4\pi \alpha_i^2} \\ &+ \sum_{j=1, j \neq i}^n 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_j (\alpha_k - \gamma_k) h_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi \alpha_i^2} v_{j\gamma}^0}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (109)$$

Hence this infinite linear system can be written in terms of a matrix $A_0^r = A_0$ with $n \times n$ infinite matrix entries. More precisely, we have

$$A_0 = \left(A_0^{ij} \right)_{1 \leq i, j \leq n}, \quad (110)$$

where for $1 \leq i, j \leq n$ we have

$$A_0^{ij} = \delta_{ij} D^0 + \delta_{ij} C^0 + L_{ij}^0 \quad (111)$$

along with the Kronecker δ -function δ_{ij} , and with the infinite matrices

$$\begin{aligned} D^0 &:= \left(-\nu \delta_{\alpha\beta} \sum_{j=1}^n \frac{4\pi \alpha_j^2}{l^2} \right)_{\alpha, \beta \in \mathbb{Z}^n}, \\ C^0 &:= \left(-\sum_{j=1}^n \frac{2\pi i \beta_j}{l} h_{j(\alpha-\beta)} \right)_{\alpha, \beta \in \mathbb{Z}^n}, \\ L_{ij}^0 &= \left((2\pi i) \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n \frac{4\pi \beta_j (\alpha_k - \beta_k) h_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi \alpha_i^2}}{\sum_{i=1}^n 4\pi \alpha_i^2} \right)_{\alpha, \beta \in \mathbb{Z}^n}, \end{aligned} \quad (112)$$

corresponding to the the Laplacian, the convection term, and the Leray projection terms respectively. Stricly speaking, we have defined a $n \times n$ matrix of infinite matrices where the Burgers equation terms are encoded on the n diagonal places and the linearized Leray projection terms exist also off diagonal. It is clear how the matrix multiplication may be defined in order to reformulate (109) and we do not dwell on these trivial formalities. We note that

$$\begin{aligned}\frac{\partial \mathbf{v}^{0,F}}{\partial t} &= A_0 \mathbf{v}^{0,F}, \\ \mathbf{v}^{0,F}(0) &= \mathbf{h}^F\end{aligned}\tag{113}$$

along with the vectors $\mathbf{v}^F = (\mathbf{v}_1^F, \dots, \mathbf{v}_n^F)^T$ and $\mathbf{h}^F = (\mathbf{h}^F, \dots, \mathbf{h}^F)^T$. Note that we have off-diagonal terms only because we consider global equations, i.e., equations, which correspond to linear partial integro-differential equations. Note that the matrices D^0, C^0, L_{ij}^0 are unbounded even for regular data h_i with fast decreasing modes (for the the matrix C^0 consider constant $\alpha - \beta$ and let β_j go to infinity for some j). The difficulty to handle unbounded infinite matrices in dual space becomes apparent. However, it is at this point that we can take advantage of the dissipative nature of the operator which is indeed the difference to the Euler equation. This is a major motivation for the dissipative Trotter product formula. Now, for two matrices $M = (m_{\alpha\beta})_{\alpha,\beta \in \mathbb{Z}^n}$ and $N = (n_{\alpha\beta})_{\alpha,\beta \in \mathbb{Z}^n}$ we may formally define the product $P = (p_{\alpha\gamma})_{\alpha,\gamma \in \mathbb{Z}^n} = MN$ via

$$p_{\alpha\gamma} = \sum_{\beta \in \mathbb{Z}^n} m_{\alpha\beta} n_{\beta\gamma} \tag{114}$$

for all $\alpha, \gamma \in \mathbb{Z}^n$. There are natural spaces for which this definition makes sense (and which we introduce below). In order to apply this natural theory of infinite matrices developped below the next step is to observe that for $k \geq 0$ matrices such as

$$\begin{aligned}\exp(D^0) (C^0)^k &= \left(\exp \left(-\nu \sum_{j=1}^n \frac{4\pi\alpha_j^2}{l^2} \right) \left(\sum_{j=1}^n \frac{2\pi i \beta_j}{l} h_{j(\alpha-\beta)} \right) \right)_{\alpha,\beta \in \mathbb{Z}^n}^k, \\ \exp(D^0) (L_{ij}^0)^k &= \left(\exp \left(-\nu \sum_{j=1}^n \frac{4\pi\alpha_j^2}{l^2} \right) \times \right. \\ &\quad \left. \left(2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi\beta_j (\alpha_k - \beta_k) h_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi\alpha_i^2} \right) \right)_{\alpha,\beta \in \mathbb{Z}^n}^k\end{aligned}\tag{115}$$

have indeed bounded entries if the modes $h_{i\beta}$ decreases sufficiently, and we shall see that with regular data h_i related infinite matrix multiplications

are well-defined. Note that it is nevertheless difficult to define the formal solution for $\mathbf{v}^{0,F}$, i.e., the expression

$$\exp((\delta_{ij}D^0 + \delta_{ij}C^0 + L_{ij}^0)t) \mathbf{h}^F \quad (116)$$

directly. For this reason we shall consider projections of infinite vectors to finite vectors with modes of order less than $l > 0$, i.e., projections P_{v^l} which are defined on infinite vectors $(g_\alpha)_{\alpha \in \mathbb{Z}^n}^T$ by

$$P_{v^l} (g_\alpha)_{\alpha \in \mathbb{Z}^n}^T := (g_\alpha)_{|\alpha| \leq l}^T \quad (117)$$

and projections P_{M^l} of infinite matrices $(m_{\alpha\beta})_{\alpha, \beta \in \mathbb{Z}^n}$ to finite matrices with

$$P_{M^l} (m_{\alpha\beta})_{\alpha, \beta \in \mathbb{Z}^n}^T := (m_{\alpha\beta})_{|\alpha| \leq l}^T \quad (118)$$

and prove Trotter product formulas for finite dissipative systems. This will lead us to the crucial observation then of a product formula of Trotter-type of the form

$$\begin{aligned} & \lim_{l \uparrow \infty} \exp\left((\delta_{ij}P_{M^l}D^0 + \delta_{ij}P_{M^l}C^0 + P_{M^l}L_{ij}^0)t\right) P_{v^l} \mathbf{h}^F \\ &= \lim_{l \uparrow \infty} \lim_{k \uparrow \infty} \left(\exp\left(\delta_{ij}P_{M^l}D^0 \frac{t}{k}\right) \exp\left(\left(\delta_{ij}P_{M^l}C^0 + P_{M^l}L_{ij}^0\right) \frac{t}{k}\right) \right)^k P_{v^l} \mathbf{h}^F, \end{aligned} \quad (119)$$

where $(\delta_{ij}D^0 + \delta_{ij}C^0 + L_{ij}^0)$, i.e., the argument of the projection operator P_{M^l} , denote 'quadratic' matrices with $(n \times \mathbb{Z}^n)$ rows and $(n \times \mathbb{Z}^n)$ columns and h is some regular function. This means that for finite $l > 0$ we can prove a Trotter type relation and in the limit the left side equals the solution $\mathbf{v}^{0,F}$ of the equation above and is defined by the right side of (119). In the last step the regularity of the data \mathbf{h}^F comes into play. In this form this formula is useful only for the stage 0 where the coefficient functions do not depend on time. For stages $m > 0$ of the construction we have to take time dependence into account. However, we may define a Euler-type scheme and define substages which produce the approximations of stage m in the limit. The formula (119) depends on an observation which uses the diagonal structure of D^0 (and therefore $\exp(D^0)$). Here the matrix $\delta_{ij}C^0 + L_{ij}^0 - \left(\delta_{ij}C^0 + L_{ij}^0\right)^T$, i.e., the deficiencies of symmetry in the matrices $\delta_{ij}C^0 + L_{ij}^0$, factorize with the correction term of an infinite analog of a special form of a CBH-type formula. It is an interesting fact that this deficiency is in a natural matrix space, we are going to define next.

Concerning the natural matrix spaces, in order to make sense of formulas as in (108) for $s > n \geq 2$ for a matrix $M = (m_{\alpha\beta})_{\alpha, \beta \in \mathbb{Z}^n}$ we say that

$$M \in M_n^s \quad (120)$$

if for all $\alpha, \beta \in \mathbb{Z}^n$

$$|m_{\alpha\beta}| \leq \frac{C}{1 + |\alpha - \beta|^{2s}} \quad (121)$$

for some $C > 0$. For $s > n$ and a vector $w = (w_\alpha)_{\alpha \in \mathbb{Z}^n} \in h^s(\mathbb{Z})$ we define the multiplication of the infinite matrix M with w by $Mw = (Mw_\alpha)_{\alpha \in \mathbb{Z}^n}$ along with

$$Mw_\alpha := \sum_{\beta \in \mathbb{Z}^n} m_{\alpha\beta} w_\beta. \quad (122)$$

Indeed we shall observe that for $r, s > n \geq 2$ we have

$$\sum_{\beta \in \mathbb{Z}^n \setminus \{0, \alpha\}} \frac{1}{|\alpha - \beta|^s \beta^r} \leq \frac{C}{1 + |\alpha|^{r+s-n}} \quad (123)$$

such that for $s > n$ and $M \in M_n^s$ we have indeed $Mw \in h^r(\mathbb{Z}^n)$ if $w \in h^r(\mathbb{Z}^n)$. This implies that for $s > n$, $M \in h^s(\mathbb{Z}^n \times \mathbb{Z}^n)$ and $w \in h^r(\mathbb{Z}^n)$

$$M^1 w := Mw, \quad M^{k+1} w = M \left(M^k \right) w \quad (124)$$

is a well defined recursion for $k \geq 1$. Hence for a matrix M which is not time-dependent the analytic vector

$$\exp(Mt) w := \sum_{k \geq 0} \frac{M^k t^k w}{k!} \quad (125)$$

is well defined (even globally). For a time dependent matrix $A = A(t)$ we formally define the time-ordered exponential 'a la Dyson' to be

$$\begin{aligned} T \exp(At) &:= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{[0,t]} dt_1 \cdots dt_m T A(t_1) \cdots A(t_m) dt_1 \cdots dt_m \\ &:= \sum_{m=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{m-1}} A(t_1) \cdots A(t_m). \end{aligned} \quad (126)$$

Note that T is the usual Dyson time-order operator which may be defined recursively using the Heavyside function. However, note that redefining with the Heavyside functions may make matters quite delicate if regularity with respect to time derivatives is considered. At each stage $m > 0$ of the construction we shall define a scheme based on the Trotter product formula such that in the limit the linear approximation expressed formally by

$$\mathbf{v}^{r,m,F}(t) := T \exp(A_m^r t) \mathbf{h}^F, \quad (127)$$

gets a strict sense. This holds also for the uncontrolled linear approximation

$$\mathbf{v}^{m,F}(t) := T \exp(A_m t) \mathbf{h}^F, \quad (128)$$

where we shall see that these expressions can be well-defined for data $h \in h^s(\mathbb{Z})$ for $s > n$ and for $t \geq 0$. Note that proving the existence of a limit of

the the iteration with respect to m in a regular space also requires $\nu > 0$. In order to prove the existence of a limit there are basically two possibilities then. One is to prove the existence of a uniform bound

$$\sup_{t \geq 0} |\mathbf{v}^{r,m,F}(t)|_{h^s} + \sup_{t \geq 0} \left| \frac{\partial}{\partial t} \mathbf{v}^{r,m,F}(t) \right|_{h^s} \leq C \quad (129)$$

for some $C > 0$ independent of m , and then proceed with a compactness arguments a la Rellich. A weaker form of (129) without the time derivative and a strong spatial norm (large s) is another variation for this alternative since product formulas for Sobolev norms with $s > \frac{1}{2}n$ and a priori estimates of Schauder type lead to an independent proof of the existence of a regular time derivative in the limit $m \uparrow \infty$. Maybe the latter variation is the most simple one. An alternative is a contraction argument on a ball of an appropriate function space. Clearly, the radius of the ball will depend on the initial data, dimension and viscosity. This dependence can be encoded in a time weight of a time-weighted norm- as it is known from ODE theory for finite equations. Contraction arguments have the advantage that they lead to uniqueness results naturally. For $(t \rightarrow w(t)) \in C([0, \infty) \times h^s(\mathbb{Z}^n))$, we define for some $C > 0$ (depending only on $\nu > 0$, the dimension $n > 0$, and the initial data components $h_i \in C^\infty(\mathbb{T}^n)$) the norm

$$|w|_{h^s,C}^{\text{exp}} := \sup_{t \in [0, \infty)} \exp(-Ct) |w(t)|_{h^s}, \quad (130)$$

and the norm

$$|w|_{h^s,C}^{\text{exp},1} := \sup_{t \in [0, \infty)} \exp(-Ct) (|w(t)|_{h^s} + |D_t w(t)|_{h^s}), \quad (131)$$

where $D_t w(t) := \left(\frac{d}{dt} w(t)_\alpha \right)_{\alpha \in \mathbb{Z}^n}^T$ denotes the vector of componentwise derivatives with respect to time t . Then a contraction property

$$\left(\delta \mathbf{v}_i^{m,F} \right)_{1 \leq i \leq n} = \left(\mathbf{v}_i^{r,m,F}(t) - \mathbf{v}_i^{r,m-1,F} \right)_{1 \leq i \leq n} \quad (132)$$

for all $1 \leq i \leq n$ with respect to both norms and for $s > n + 2$ and $1 \leq i \leq n$ can be proved. Summarizing we have the following steps:

- i) in the first step we do some matrix analysis. First, the multiplication of infinite matrices A_0 and A_m with infinite vectors \mathbf{h}^F is well defined. Second matrix multiplication in the matrix space M_n^s is well-defined for $s > n \geq 2$ as is $\exp(M)$ for $M \in M_n^s$. Then we prove a dissipative Trotter product formula for finite systems and apply the result to the first infinite approximation system with solution $\mathbf{v}^{0,F}$ at stage $m = 0$ which is related to a linear integro-partial differential equation.

- ii) In the second step we set up an Euler-type scheme based on the Trotter-product formula which shows that for some $\nu > 0$ and $s > n \geq 2$ the exponential

$$\mathbf{v}_i^{r,m,F}(t) := (T \exp(A_m^r t) \mathbf{h}^F)_i \quad (133)$$

is well-defined for all $1 \leq i \leq n$, and in $h_l^s(\mathbb{Z}^n)$ for all m , i.e., that the linearized equations for $\mathbf{v}_i^{r,m,F}$ which are equivalent to a linear partial integro-differential equation have a global solution. Here $(\cdot)_i$ indicates that we project to the i th component of the n infinite entries of the solution vector at the first stage of construction.

- iii) for some $\nu > 0$ as in step ii) the limit

$$\mathbf{v}_i^{r,F}(t) := (T \exp(A_\infty^r t) \mathbf{h}^F)_i \quad (134)$$

exists, where

$$\begin{aligned} \mathbf{v}_i^{r,F}(t) &:= \lim_{m \uparrow \infty} \mathbf{v}_i^{r,m,F}(t) \\ &= (T \exp(A_\infty^r t) \mathbf{h}^F)_i := \lim_{m \uparrow \infty} (T \exp(A_m^r t) \mathbf{h}^F)_i \in h_l^s(\mathbb{Z}^n). \end{aligned} \quad (135)$$

This limit can be obtained by compactness arguments and by a contraction results in time-weighted regular function spaces. The latter result leads to uniqueness of solutions in the time-weighted function space.

Concerning the third step (iii) we note that for all $t \geq 0$ and $x \in \mathbb{T}_l^n$ we have

$$v_j^{r,m}(t, x) := \sum_{\alpha \in \mathbb{Z}^n} v_{i\alpha}^{r,m,F}(t) \exp(2\pi i \alpha x) \quad (136)$$

for all $1 \leq j \leq n$. Hence, in classical Sobolev function spaces we have a compact sequence $(v_j^{r,m,F}(t, \cdot))_{m \in \mathbb{N}}$ in higher order Sobolev spaces H^r with $r > s$ by the Rellich embedding theorem on compact manifolds for fixed $t \geq 0$. In this context recall that

Theorem 2.1. *For any $q > s$, $q, s \in \mathbb{R}$ and any compact manifold M the embedding*

$$j : H^q(M) \rightarrow H^s(M) \quad (137)$$

is compact.

For any $t \geq 0$ we have a limit $v_j^r(t, \cdot) \in H^s(\mathbb{T}^n) \subset C^m$ for $s > m + \frac{n}{2}$ and the fact that the control function r is well-defined and continuous implies

$$v_j(t, \cdot) = v_j^r(t, \cdot) - r(t) \in C^m \quad (138)$$

for fixed $t \geq 0$ and all $m \in \mathbb{N}$. Finally we verify that v_j is indeed a classical solution of the original Navier-Stokes equation using the properties we proved for the vector $v^{r,F}$ in the dual formulation. One possibility to do this is to plug in the approximations of the solutions and estimate the remainder pointwise observing that it goes to zero in the strong norm $|w|_{h^s}^{\text{exp},1}$.

3 Proof of theorem 1.2

We consider data $h_i \in C^\infty(\mathbb{T}_l^n)$ for $1 \leq i \leq n$. The special domain of a n -torus has the advantage that we may represent approximations of a solution evaluated at a given time $t \geq 0$ in the form of Fourier mode coefficients with respect to the basis

$$\left\{ \exp\left(\frac{2\pi i \alpha x}{l}\right) \right\}_{\alpha \in \mathbb{Z}^n}, \quad (139)$$

i.e., with respect to an orthonormal basis of $L^2(\mathbb{T}_l^n)$. It is natural to start with an expansion of the solution $\mathbf{v} = (v_1, \dots, v_n)^T$ in the form

$$\begin{aligned} v_i(t, x) &= \sum_{\alpha \in \mathbb{Z}^n} v_{i\alpha} \exp\left(\frac{2\pi i \alpha x}{l}\right), \\ p(t, x) &= \sum_{\alpha \in \mathbb{Z}^n} p_\alpha \exp\left(\frac{2\pi i \alpha x}{l}\right), \end{aligned} \quad (140)$$

where the modes $v_{i\alpha}$, $\alpha \in \mathbb{Z}^n$ and p_α , $\alpha \in \mathbb{Z}^n$ depend on time. Here we have an index $1 \leq i \leq n$ which gives rise to some ambiguity with the complex number $2\pi i$, but the latter i always occurs in the context of π such that no confusion should arise. Plugging this ansatz into the Navier-Stokes equation system (76) we formally get n coupled infinite differential equations of ordinary type for the modes, i.e., for all $1 \leq i \leq n$ we have for all $\alpha \in \mathbb{Z}^n$

$$\begin{aligned} \frac{dv_{i\alpha}}{dt} &= \nu \sum_{j=1}^n \left(-\frac{4\pi \alpha_j^2}{l^2} \right) v_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)} v_{i\gamma} \\ &\quad - 2\pi i \alpha_i p_\alpha, \end{aligned} \quad (141)$$

where for each $1 \leq i \leq n$ and each $\alpha \in \mathbb{Z}^n$

$$v_{i\alpha} : [0, \infty) \rightarrow \mathbb{R} \quad (142)$$

are some time-dependent α -modes of velocity, and

$$p_\alpha : [0, \infty) \rightarrow \mathbb{R} \quad (143)$$

are some time-dependent α -modes of pressure to be determined in terms of the velocity modes $v_{i\alpha}$. Here, $(\alpha - \gamma)$ denotes the subtraction between multiindices, i.e.,

$$(\alpha - \gamma) = (\alpha_1 - \gamma_1, \dots, \alpha_n - \gamma_n), \quad (144)$$

where brackets are added for notational reasons in order to mark separate multiindices. Next we may eliminate the pressure modes p_α using Leray projection, which shows that the pressure p satisfies the Poisson equation

$$\Delta p = - \sum_{j,k} v_{j,k} v_{k,j}, \quad (145)$$

where we use the Einstein abbreviation for differentiation, i.e.,

$$v_{j,k} = \frac{\partial v_j}{\partial x_k} \text{ etc..} \quad (146)$$

The Poisson equation in (145) is reexpressed by an infinite equation system for the α -modes of the form

$$p_\alpha \sum_{i=1}^n \frac{-4\pi^2 \alpha_i^2}{l^2} = \sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{l^2} \quad (147)$$

with the formal solution (w.l.o.g. $\alpha \neq 0$ -see below)

$$p_\alpha = -1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}, \quad (148)$$

for $\alpha \neq 0$, which is indeed independent of the size of the torus l . This means that this term becomes large compared to the second order terms and the convection term for large tori ($l > 0$ large while viscosity $\nu > 0$ stays fixed). This may happen for approximations of Cauchy problems by equations for large tori for purpose of simulation. Note that

$$1_{\{\alpha \neq 0\}} := \begin{cases} 1 & \text{if } \alpha \neq 0, \\ 0 & \text{else.} \end{cases} \quad (149)$$

Note that we put $p_0 = 0$ for $\alpha = 0$ in (148). We are free to do so since $(\mathbf{v}, p + C)$ is a solution of (76) if (\mathbf{v}, p) is. Plugging into (141) we get for all $1 \leq i \leq n$, and all $\alpha \in \mathbb{Z}^n$

$$\begin{aligned} \frac{dv_{i\alpha}}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi \alpha_j^2}{l^2} \right) v_{i\alpha} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)} v_{i\gamma} \\ &+ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma} v_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \end{aligned} \quad (150)$$

For $1 \leq i \leq n$, this is a system of nonlinear ordinary differential equations of 'infinite dimension', i.e., for infinite modes $v_{i\alpha}$ and p_α along with $\alpha \in \mathbb{Z}^n$. We emphasize that it may be rewritten in the basis

$$\cos(\alpha x), \quad \sin(\alpha x), \quad \alpha \in \mathbb{Z}^n, \quad (151)$$

with real modes at every step. Take (150) as a shorthand notation for an equivalent equation with respect to a real basis as in (151). This implies that we are constructing real solutions, although our notation is complex. For computations this real basis may be further reduced (cf. section on computational issues below). This means that we construct real solutions $v_{i\alpha}$ with $v_{i\alpha}(t) \in \mathbb{R}$. Note that it suffices to prove that there is a regular solution to (150), because this translates into a classical regular solution of the incompressible Navier-Stokes equation on the n -torus, and this implies the existence of a classical solution of the incompressible Navier-Stokes equation in its original formulation, i.e., without elimination of the pressure. We provided an argument of this well-known implication at the end of section 1. This is not the controlled scheme which we considered in the first section. However, this scheme is identical with the controlled scheme at the first stage $m = 0$, so we start with some general considerations which apply to this first stage first and introduce the control function later. So let us look at the simple scheme in more detail now. The first approximation is the system

$$\begin{aligned} \frac{dv_{i\alpha}^0}{dt} = & \sum_{j=1}^n \left(-\nu \frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^0 - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} h_{j(\alpha-\gamma)} v_{i\gamma}^0 \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^0 h_{k(\alpha-\gamma)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \end{aligned} \quad (152)$$

Let

$$\mathbf{v}^{0,F} = (\mathbf{v}_1^{0,F}, \dots, \mathbf{v}_n^{0,F}), \quad \mathbf{v}_i^{0,F} := (v_{i\alpha}^0)_{\alpha \in \mathbb{Z}^n}^T. \quad (153)$$

Formally, we consider $\mathbf{v}_i^{0,F}$ as an infinite vector, where the upper script T in the componentwise description indicates transposition. Then equation (152) may be formally rewritten in the form

$$\frac{d\mathbf{v}_i^{0,F}}{dt} = A_0^i \mathbf{v}_i^{0,F} + \sum_{j \neq i} L_{ij}^0 \mathbf{v}_j^{0,F}, \quad (154)$$

with the infinite matrix $A_0^i = (a_{\alpha\beta}^{i0})_{\alpha, \beta \in \mathbb{Z}^n}$ and the entries

$$a_{\alpha\beta}^{i0} = \delta_{\alpha\beta} \nu \left(-\sum_{j=1}^n \frac{4\pi\alpha_j^2}{l^2} \right) - \sum_{j=1}^n \frac{2\pi i \beta_j h_{j(\alpha-\beta)}}{l} + L_{ii\alpha\beta}, \quad (155)$$

and where

$$L_{ij\alpha\beta}^0 = 1_{\{\alpha \neq 0\}} 2\pi i \alpha_i \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) h_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2} \quad (156)$$

for all $1 \leq i, j \leq n$ denote coupling terms related to the Leray projection term. As we observed in the preceding section you may write (154) in the form

$$\frac{d\mathbf{v}^{0,F}}{dt} = A_0 \mathbf{v}^{0,F}, \quad (157)$$

where

$$A_0 \mathbf{v}^{0,F} := \left(\left(\sum_{j \in \{1, \dots, n\}, \beta \in \mathbb{Z}^n} ((\delta_{ij} a_{\alpha\beta}^{i0}) + L_{ij\alpha\beta}^0) v_{j\beta}^0 \right)_{\alpha \in \mathbb{Z}^n} \right)_{1 \leq i \leq n}^T, \quad (158)$$

and $A_0 = (A_0^{ij})$ can be considered as a quadratic matrix with n^2 infinite matrix entries

$$A_0^{ij} = A_0^i \text{ for } i = j, \quad (159)$$

and

$$A_0^{ij} = L_{ij}^0 \text{ for } i \neq j. \quad (160)$$

Note that these matrices which determine the first linear approximation matrix A_0 are not time-dependent. The modes $a_{\alpha\beta}^0, v_{i\beta}^0$, or at least one set of these modes, have to decay appropriately as $|\alpha|, |\beta| \uparrow \infty$ in order that the definition in (158) makes sense, i.e., leads to finite results in appropriate norms which show that the infinite set of modes in $A_0 \mathbf{v}^{0,F}$ belong to a regular function in classical space. Since the matrix A_0 has constant entries, the formal solution of (154) is

$$\mathbf{v}^{0,F} = \exp(A_0 t) \mathbf{h}^F. \quad (161)$$

In order to make sense of this formula we shall use the dissipative feature on the diagonal terms and a Trotter-type product formula (otherwise we are in trouble because the modulus of entries increases with the order of the modes). We solve the equation (264) by approximations via systems of finite modes, i.e., via

$$P_{v^l} \mathbf{v}_i^{0,F;*} = \exp(P_{M^l} A_0 t) P_{v^l} \mathbf{h}^F \quad (162)$$

where P_{v^l} and P_{M^l} denote projections of vectors and matrices to finite cut-off vectors and finite cut-off matrices of modes of order less or equal to l . Note that we used the notation

$$P_{v^l} \mathbf{v}_i^{0,F;*} \quad (163)$$

with a star superscript since the projection of the solution of the infinite system is not equal to the solution of the projected system in general. The latter is an approximation of the former which becomes equal in the limit. Indeed we use

$$\begin{aligned} \mathbf{v}_i^{0,F} &= \lim_{l \uparrow \infty} P_{v^l} \mathbf{v}_i^{0,F;*} = \lim_{l \uparrow \infty} \lim_{k \uparrow \infty} \left(\exp(P_{M^l} (\delta_{ij} D^0) \frac{t}{k}) \times \right. \\ &\quad \left. \times \exp\left(\left(P_{M^l} (\delta_{ij} C^0 + L_{ij}^0)\right) \frac{t}{k}\right) \right)^k P_{v^l} \mathbf{h}_i^F, \end{aligned} \quad (164)$$

Note that $\exp((\delta_{ij}D^0)\frac{t}{k})\exp\left(\left((\delta_{ij}C^0 + L_{ij}^0)\right)\frac{t}{k}\right)_{ij} \in M_n^s$. Here the symbol $(\cdot)_{ij}$ indicates projection onto the infinite $\mathbb{Z}^n \times \mathbb{Z}^n$ -block at the i th row and the j th column of the matrix A_0 . This means that the right side is well defined due to the matrix $\exp((\delta_{ij}D^0)\frac{t}{k})$ which has the effect of an multiplication of each row by a function which is exponentially decreasing with respect to time and with respect to the order of the modes. The more delicate thing is to prove that the Trotter-type approximation converges to the (approximative) solution at each stage m of the construction. We come back to this later in the proof of the dissipative Trotter product formula. Let us start with the description of the other stages $m > 0$ of the construction first. At stage $m \geq 1$ having computed $\mathbf{v}_i^{m-1,F} = (v_{i\alpha}^{m-1})_{\alpha \in \mathbb{Z}^n}^T$ the m th approximation is computed via the system

$$\begin{aligned} \frac{dv_{i\alpha}^m}{dt} = & \sum_{j=1}^n \left(-\frac{4\pi\alpha_j^2}{l^2}\right) v_{i\alpha}^m - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{m-1} v_{i\gamma}^m \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^m v_{k(\alpha-\gamma)}^{m-1}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2} \end{aligned} \quad (165)$$

with the same initial data, of course. We have

$$\mathbf{v}_i^{m,F} := (v_{i\alpha}^m)_{\alpha \in \mathbb{Z}^n}^T, \quad (166)$$

and the equation (165) may be formally rewritten in the form

$$\frac{d\mathbf{v}^{m,F}}{dt} = A_m \mathbf{v}^{m,F}, \quad (167)$$

with the infinite matrix $A_m = (a_{\alpha\beta}^{ijm})_{\alpha,\beta \in \mathbb{Z}^n}$ and the (time-dependent!) entries $A_m = (A_m^{ij}) = (a_{\alpha\beta}^{ijm})$, and along with $A_m^{ii} = D^0 + C_{ii}^m + L_{ii}^m$ for all $1 \leq i \leq n$, and $A_m^{ij} = L_{ij}^m$ for $i \neq j$, $1 \leq i, j \leq n$, where for all $i \neq j$ we have

$$L_{ij\alpha\beta}^m = 2\pi i \alpha_i \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^{m-1}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \quad (168)$$

The matrix C_{ii}^m related to the convection term $-\sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{m-1} v_{i\gamma}^m$ is defined analogously as in the first stage but with coefficients $v_{i\alpha}^{m-1}$ instead of $h_{i\alpha}$. Again, the equation (166) makes sense for regular data and inductively assumed regular coefficient functions v_i^{m-1} , because polynomially decay of the modes $h_{i\alpha}$ and $v_{i\alpha}$ compensate that encoded quadratic growth due to derivatives up to second order. For a solution however we have to make sense of exponential functions with exponent $a_{\alpha\beta}^{ijm}$ defined in terms of modes $v_{i\beta}^{m-1}$ and applied to data \mathbf{h}^F . Again we shall use a dissipative Trotter product formula in order to have appropriate decay as $|\alpha|, |\beta| \uparrow \infty$

in order for the solution $v_{i\alpha}^m$. However, this time we have time-dependent coefficients and we need a subscheme at each stage m of the construction in order to deal with time dependent coefficients. The formal solution of (167) is

$$\mathbf{v}^{m,F} = T \exp(A_m t) \mathbf{h}^F, \quad (169)$$

where $T \exp(\cdot)$ is a Dyson-type time-order operator T defined above. Note that for all $m \geq 1$ the functions $\mathbf{v}_i^{m,F}$, $1 \leq i \leq n$ represent formal solutions of partial integro-differential equation if we rewrite them in the original space. Again in order to make sense of them we shall use the dissipative feature and a Trotter-type product formula at each substage which is a natural time-discretization. It seems that even at this stage we need viscosity $\nu > 0$ in order to obtain solutions for these linear equation in this dual context. This assumption is also needed when we consider the limit $m \uparrow \infty$. We shall estimate at each stage m of the construction

$$\mathbf{v}^{m,F} = T \exp(A_m t) \mathbf{h}^F \quad (170)$$

based on the Trotter product formula where we set up an Euler-type scheme in order to deal with time-dependent infinite matrices in the limit of sub-stages at each stage m .

Next we go into the details of this plan. Let us consider some linear algebra of time-independent infinite matrices with fast decaying entries (which can be applied directly at the stage $m = 0$). The matrices of the scheme considered above are $(n \times \mathbb{Z}^n) \times (n \times \mathbb{Z}^n)$ -matrices, but - for the sake of simplicity of notation - we shall consider some matrix-algebra for $\mathbb{Z}^n \times \mathbb{Z}^n$ -matrices. The considerations can be easily adapted to the formally more complicated case. First let $D = (d_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n}$ and $E = (e_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n}$ be two infinite matrices, and define (formally)

$$D \cdot E = (f_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n}, \quad (171)$$

where

$$f_{\alpha\beta} = \sum_{\gamma \in \mathbb{Z}^n} d_{\alpha\gamma} e_{\gamma\beta}. \quad (172)$$

Next we define a space of matrices such that (171) makes sense. For $s \in \mathbb{R}$ we define

$$M_n^s := \left\{ D = (d_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n} \mid d_{\alpha\beta} \in \mathbb{C} \ \& \ \exists C > 0 : |d_{\alpha\beta}| \leq \frac{C}{1 + |\alpha - \beta|^s} \right\}. \quad (173)$$

In the following we consider rather regular spaces where $s \geq 2 + n$. Some results can be optimized with respect to regularity, but our purpose here is full regularity in the end. Next we have

Lemma 3.1. *Let $D \in M_n^s$ and $E \in M_n^r$ for some $s, r \geq n + 2$. Then*

$$D \cdot E \in M_n^{r+s-n} \quad (174)$$

Proof. For some $c > 0$ we have for $\alpha \neq \beta$

$$\begin{aligned}
|f_{\alpha\beta}| &= \left| \sum_{\gamma \in \mathbb{Z}^n} d_{\alpha\gamma} e_{\gamma\beta} \right| \\
&\leq c + \sum_{\gamma \notin \{\alpha, \beta\}} \frac{C}{|\alpha - \gamma|^s} \frac{C}{|\gamma - \beta|^r} \\
&\leq c + \frac{cC}{|\alpha - \beta|^{r+s-n}}
\end{aligned} \tag{175}$$

The latter inequality is easily obtained by comparison with the integral

$$\int_{\mathbb{R}^n \setminus B_{\alpha\beta}} \frac{dy}{|\alpha - y|^r |y - \beta|^s} \tag{176}$$

where $B_{\alpha\beta}$ is the union of balls of radius $1/2$ around α and around β . Partial integration of these integrals in polar coordinate form in different cases leads to the conclusion. We observe here that there is some advantage here in the analysis compared to the analysis in classical space because we can avoid the analysis of singularities in discrete space. \square

For $r = s \geq n$ this behavior allows us to define iterations of matrix multiplications recursively according to the matrix rules defined in (171) and (172), i.e., we may define by recursion

$$\begin{aligned}
A^0 &= I, \text{ where } I = (\delta_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n}, \\
A^1 &= A, \\
A^{k+1} &= A \cdot A^k.
\end{aligned} \tag{177}$$

In the matrix space M_n^s we may define exponentials. For our problem this space is too 'narrow' to apply this space directly. However, it is useful to note that we have exponentials in this space.

Corollary 3.2. *Let $D \in M_n^s$ for some $s \geq n$. Then*

$$\exp(D) = \sum_{k=0}^{\infty} \frac{D^k}{k!} \in M_n^s \tag{178}$$

Well what we need is an estimate for the approximative solutions. In this context we note

Corollary 3.3. *Let $D \in M_n^s$ for some $s \geq 2n$ and let $\mathbf{h}^F \in h^s(\mathbb{Z}^n)$. Then*

$$\exp(D)\mathbf{h}^F \in h^s(\mathbb{Z}^n). \tag{179}$$

Proof. Let

$$F = \exp(D), \text{ where } F = (f_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n}, \quad (180)$$

and let

$$g_\alpha = \sum_{\beta \in \mathbb{Z}^n} f_{\alpha\beta} h_\beta. \quad (181)$$

Then for some $C, c > 0$

$$|g_\alpha| = \left| \sum_{\gamma \in \mathbb{Z}^n} f_{\alpha\gamma} h_\gamma \right| \leq \sum_{\beta \in \mathbb{Z}^n \setminus \{\alpha, 0\}} \frac{C}{|\alpha - \beta|^s |\beta|^s} \leq \frac{cC}{1 + |\alpha|^s}. \quad (182)$$

□

We cannot apply the preceding lemmas directly due to the fact that the diagonal matrix with entries $-\nu \delta_{ij} \sum_{k=1}^n \frac{4\pi^2}{l^2} \alpha_k^2$ is not bounded. Neither is the the matrix related to the convection term. However the multiplication of the dissipative exponential with iterative multiplications of the convection term matrix stay in a regular matrix space such that a multiplication with regular data of polynomial decay lead to regular results. The dissipative term - the smoothing effect of which is obvious in classical space - makes the difference. At first glance, iterations of the matrix lead to matrices which live in weaker and weaker spaces, and we really need the dissipative feature, i.e., the minus signs in the diagonal in mathematical terms, in order to detect the smoothing effect in the exponential form. The irregularity related to a positive sign reminds us of the fact that heat equations cannot be solved backwards in general. However, due to its diagonal structure and its negative sign we can prove BCH-type formula for infinite matrices. The following has some similarity with Kato's results for semigroups of dissipative operators (cf. [14]). However Kato's results (cf. [14]) usually require that the domain of the dissipative operator includes the domain of the second operator summand, and this is not true in our formulation for the incompressible Navier-Stokes equation, because the diagonal operator related to the Laplacian, i.e., the diagonal terms $(-\delta_{\alpha\beta} \nu \sum_{i=1}^n \alpha_i^2)_{\alpha, \beta \in \mathbb{Z}^n \setminus \{0\}}$, increase quadratically with the order of the modes α . So it seems that the cannot be applied directly. Anyway we have

Lemma 3.4. *Let g_l^F be finite vectors of modes of order less or equal to $l > 0$. Then for some finite vector $f_l^F = (f_\alpha^l)_{|\alpha| \leq l}$ with finite entries f_α^l , and in the situation of Lemma 3.7 we have*

$$\left| \left(\exp(C^l t) \mathbf{g}_l^F - \exp((A^l + B^l)t) \mathbf{g}_l^F \right)_\alpha \right| \leq |f_\alpha^l t^2|, \quad (183)$$

where $(\cdot)_\alpha$ denotes the projection to the α th component of an infinite vector.

This is what we need in the following but let us have a closer look at the Trotter product formula. First we define

Definition 3.5. A diagonal matrix $(\delta_{\alpha\beta}d_{\alpha\beta})_{\alpha,\beta \in \mathbb{Z}^n}$ is called strictly dissipative of order $m > 0$ if $d_{\alpha\alpha} < -c|\alpha|^m$ for all $\alpha \in \mathbb{Z}^n$ and for some constant $c > 0$. It is called dissipative if it is dissipative of some order m .

In order to state the Trotter-product type result we introduce some notation.

Definition 3.6. For all $m \geq 1$ let $\gamma^m = (\gamma_1^m, \dots, \gamma_m^m)$ denote multiindices with m components and with nonnegative integer entries γ_i^m for $m \geq 1$ and $1 \leq i \leq m$. For each $m \geq 1$ we denote the set of m -tuples with nonnegative entries by \mathbb{N}_0^m . Let $B = (b_{\alpha\beta})_{\alpha,\beta \in \mathbb{Z}^n}$, denote a quadratic matrix, and let $B^T = (b_{\beta\alpha})_{\alpha,\beta \in \mathbb{Z}^n}$ be its transposed, and E be some other infinite matrix of the same type. For $m \geq 1$ and m -tuples $\gamma^m = (\gamma_1^m, \dots, \gamma_m^m)$ with nonnegative entries γ_j^m , $1 \leq j \leq m$, we introduce some abbreviations for certain iterations of Lie brackets operations of matrices. These are iterations of the matrix $\Delta B := B - B^T$ and either the matrix B or the matrix B^T in arbitrary order. This gives different expressions dependent on the matrix with which we start, and we define I_{γ^m} (starting with $[\Delta B, B]$) and $I_{\gamma^m}^T$ (starting with $[\Delta B, B^T]$) accordingly. First we define $I_{\gamma_1^1}(\Delta B, B) = I_{(\gamma_1^1)}(\Delta B, B)$ (starting with $[\Delta B, B^T]$ for $\gamma_1 \geq 0$ recursively). Let

$$[E, B]_T = EB - B^T E, \quad (184)$$

which is a Lie-bracket type operation with the transposed. For $\gamma_1^1 = 0$ define

$$I_{(\gamma_1^1)}[\Delta B, B] := \Delta B, \quad (185)$$

and for $\gamma_1^1 > 0$ define

$$I_{(\gamma_1^1)}[\Delta B, B] := [I_{\gamma_1-1}[\Delta B, B], B]_T. \quad (186)$$

Having defined $I_{\gamma^{m-1}}$ and if $\gamma_m^m = 0$ then

$$I_{\gamma^m}[\Delta B, B] = I_{\gamma^{m-1}}[\Delta B, B] + I_{\gamma^{m-1}}[\Delta B, B^T] \quad (187)$$

Finally, if $\gamma_m^m > 0$, then define

$$I_{\gamma^m}[\Delta B, B] = [I_{\gamma^{m-1}}[\Delta B, B], B]_T. \quad (188)$$

Similarly, for $\gamma_1^1 = 0$ define

$$I_{(\gamma_1^1)}^T[\Delta B, B] := \Delta B, \quad (189)$$

and for $\gamma_1^1 > 0$ define

$$I_{(\gamma_1^1)}^T[\Delta B, B] := [I_{\gamma_1-1}[\Delta B, B^T], B]_T. \quad (190)$$

Having defined $I_{\gamma^{m-1}}$ and if $\gamma_m^m = 0$ then

$$I_{\gamma^m}^T [\Delta B, B] = I_{\gamma^{m-1}}^T [\Delta B, B] + I_{\gamma^{m-1}}^T [\Delta B, B^T] \quad (191)$$

Finally, if $\gamma_m^m > 0$, then define

$$I_{\gamma^m}^T [\Delta B, B] = \left[I_{\gamma^{m-1}}^T [\Delta B, B], B^T \right]_T. \quad (192)$$

Next we prove a special CBH-formula for finite matrices. We do not need the full force of the lemma 3.7 below for our purpose, but it has some interest of its own. The reader who is interested only in the global existence proof may skip it and consider the simplified alternative considerations in order to see that how a Trotter product result can be applied. The results may be generalized but our main purpose in this article is to define a converging algorithm for the incompressible Navier-Stokes equation which provides also a constructive approach to global existence. We show

Lemma 3.7. *Define the set of finite modes*

$$\mathbb{Z}_l^n := \{\alpha \in \mathbb{Z}^n \mid |\alpha| \leq l\}. \quad (193)$$

Let A^l be the cut-off of order l of the dissipative diagonal matrix of order 2 related to the Laplacian and let $B^l = (b_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}_l^n}$ be the cut-off of some other matrix. Next for an arbitrary finite quadratic matrix $N^l = (n_{\alpha\beta})_{|\alpha|, |\beta| \leq l}$ let

$$\exp_m(N) = \exp(N) - \left(\sum_{k=0}^{m-1} \frac{N^k}{k!} \right). \quad (194)$$

Then the relation

$$\exp(A^l) \exp(B^l) = \exp(C^l), \quad (195)$$

holds where C^l is of the form

$$\begin{aligned} C^l = & A^l + B^l + \frac{1}{2} \exp(2A^l) \Delta B^l + \sum_{m \geq 1} \exp_m(A^l) \times \\ & \times \left(c_{\beta^m} I_{\beta^m} [\Delta B^l, B^l]_T + c_{\beta^m}^T I_{\beta^m}^T [\Delta B^l, B^l]_T \right) \end{aligned} \quad (196)$$

for some constants $c_{\beta^m}, c_{\beta^m}^T$ such that the series

$$\sum_{m \geq 1} c_{\beta^m} t^m \quad (197)$$

and the series

$$\sum_{m \geq 1} c_{\beta^m}^T t^m \quad (198)$$

converges absolutely for all $t \geq 0$.

Proof. For simplicity of notation we suppress the upper script l in the following. All matrices A, B, C, \dots are finite multiindexed matrices of modes of order less or equal to l . The matrix $C = \sum_{i=1}^{\infty} C_i$ is formally determined via power series

$$C(x) = \sum_{i=1}^{\infty} C_i x^i, \quad C'(x) = \sum_{i=1}^{\infty} i C_i x^{i-1} \quad (199)$$

by the relation

$$\sum_{k=0}^{\infty} R^k [C'(x), C(x)] = A + B + \sum_{k \geq 1} R^k [A, B] \frac{x^k}{k!}, \quad (200)$$

where for matrices E, F $[E, F] = EF - FE$ denotes the Lie bracket and

$$R^1[E, F] = [E, F], \quad R^{k+1}[E, F] = [R^k[E, F], F] \quad (201)$$

recursively (Lie bracket iteration on the right). Comparing the terms of different orders one gets successively for the first C -terms

$$\begin{aligned} C_1 &= A + B, \quad C_2 = \frac{1}{2} [A, B], \\ C_3 &= \frac{1}{12} [A, [A, B]] + \frac{1}{12} [[A, B], B], \\ C_4 &= \frac{1}{48} [A, [[A, B], B]] + \frac{1}{48} [[A, [A, B]], B] \\ C_5 &= \frac{1}{120} [A, [[A, [A, B]], B]] + \frac{1}{120} [A, [[[A, B], B], B]] \\ &\quad - \frac{1}{360} [A, [[[A, B], B], B]] - \frac{1}{360} [[A, [A, [A, B]]], B] \\ &\quad - \frac{1}{720} [A, [A, [A, [A, B]]]] - \frac{1}{720} [[[[A, B], B], B], B], \dots \end{aligned} \quad (202)$$

Iterated Lie brackets simplify if A is a diagonal matrix. First we define left Lie-bracket iterations, i.e.,

$$L^1[E, F] = [E, F], \quad L^{k+1}[E, F] = [E, [L^k[E, F]]] \quad (203)$$

recursively. Next we study the effect of alternative applications of iterations of the left and right Lie-bracket operation for this specific A . We have

$$[A, B] = A \Delta B, \quad (204)$$

with $\Delta B = B - B^T$, and

$$L^k [A, B] = A^k 2^{k-1} \Delta B, \quad (205)$$

Next we have

$$R^k[A, B] = AR^{k-1}[\Delta B, B]_T \quad (206)$$

Next

$$LR^k[A, B] = A^2R^{k-1}([\Delta B, B]_T + [\Delta B, B^T]_T) \quad (207)$$

Induction leads to the observation that other summands than $A + B$ of the series for C can be written in terms of expressions of the form

$$(A)^k I_{\beta^m} [\Delta B, B]_T \quad (208)$$

and in terms of expressions of the form

$$(A)^k I_{\beta^m}^T [\Delta B, B]_T \quad (209)$$

For each order $p = k + m$ we have a factor $\frac{1}{p!}$ with $\leq 2^p$ summands. This leads to global convergence of the coefficients $c_{\beta^m}, c_{\beta^m}^T$. \square

We continue to describe consequences in a framework which is very close to the requirements of the systems related to the incompressible Navier-Stokes equation. As a simple consequence of the preceding lemma we have

Lemma 3.8. *Let g_l^F be a finite vector of modes of order less or equal to $l > 0$. In the situation of Lemma 3.7 we have*

$$\lim_{k \uparrow \infty} \left(\exp \left(A^l \frac{t}{k} \right) \exp \left(B^l \frac{t}{k} \right) \right)^k g_l^F = \exp \left((A^l + B^l)t \right) g_l^F. \quad (210)$$

It is rather straightforward to reformulate the latter result to equations with matrices of type $P_{M^l} A_0, P_{M^l} A_0^r$, i.e. finite approximations of order l of the $n\mathbb{Z}^n \times n\mathbb{Z}^n$ -matrices A_0, A_0^r . We have

Corollary 3.9. *Let $B_b^{0,l} = \left(P_{M^l} \left(C_i^0 + L_{ij}^0 \right)_{ij} \right)$ and $B_b^{r,l,0} = \left(P_{M^l} \left(C_i^{r,0} + L_{ij}^{r,0} \right)_{ij} \right)$ such that $A_0^l = D_b^{0,l} + B_b^{0,l}, A_0^{r,l} = D_b^{0,l} + B_b^{r,l}$. Then the Trotter product formula*

$$\lim_{k \uparrow \infty} \left(\exp \left(D_b^{0,l} \frac{t}{k} \right) \exp \left(B_b^{0,l} \frac{t}{k} \right) \right)^k = \exp \left(A_0^l t \right), \quad (211)$$

and the Trotter product formula

$$\lim_{k \uparrow \infty} \left(\exp \left(D_b^{r,l,0} \frac{t}{k} \right) \exp \left(B_b^{r,l,0} \frac{t}{k} \right) \right)^k = \exp \left(A_0^{r,l} t \right) \quad (212)$$

hold.

Next note that iterations of $B_b^{0,l}$ of order $k \geq 1$ have at most linear growth for constellations of multiindexes where $\alpha - \gamma$ is constant (with the order of the modes as $l \uparrow \infty$). Let us have a closer look at this. We have

$$B_{b,ij\alpha\beta}^{0,l} = -\sum_{j=1}^n \frac{2\pi i \beta_j}{l} h_{j(\alpha-\beta)} + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) h_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \quad (213)$$

We consider the entries for the square. We have

$$\begin{aligned} \sum_{\beta \in \mathbb{Z}^n} B_{b,ij\alpha\beta}^{0,l} B_{b,ij\beta\gamma}^{0,l} = \\ \left(-\sum_{j=1}^n \frac{2\pi i \beta_j}{l} h_{j(\alpha-\beta)} + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) h_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2} \right) \\ \left(-\sum_{j=1}^n \frac{2\pi i \gamma_j}{l} h_{j(\beta-\gamma)} + 2\pi i \beta_i 1_{\{\beta \neq 0\}} \frac{\sum_{k=1}^n 4\pi \gamma_j (\beta_k - \gamma_k) h_{k(\beta-\gamma)}}{\sum_{i=1}^n 4\pi^2 \beta_i^2} \right) \end{aligned} \quad (214)$$

Expanding the latter product (214) and inspecting the four terms of the expansion we observe that two of these four terms are entries of matrices in the regular matrix space. Only the mixed products of convection terms entries and Leray projection terms entries lead to expressions which are a little less regular. We define an appropriate matrix space

$$M_n^{s,\text{lin}} := \left\{ D = (d_{\alpha\beta})_{\alpha\beta \in \mathbb{Z}^n} \mid d_{\alpha\beta}^l \in \mathbb{C} \ \& \ \exists C > 0 \ \forall l : |d_{\alpha\beta}^l| \leq \frac{C+C(|\alpha|+|\beta|)}{1+|\alpha-\beta|^s} \right\}. \quad (215)$$

By induction we have

Lemma 3.10. *Given $1 \leq i, j \leq n$ for all $k \geq 1$ we have*

$$\left(B_{b,ij\alpha\beta}^{0,l} \right)^k \in M_n^{s,\text{lin}} \quad (216)$$

Hence the statement of lemma 3.10 is true also for the exponential of $B_{b,ij}^{0,l}$. We note

Lemma 3.11. *If D is a strictly dissipative diagonal matrix of order 2, then for given $1 \leq i, j \leq n$ we have*

$$\exp(D) \exp \left(B_{b,ij\alpha\beta}^{0,l} \right) \in M_n^s. \quad (217)$$

In order to establish Trotter product representations for our iterative linear approximations $v^{r,m,F}$ of a controlled Navier-Stokes equation we consider a matrix space for sequences of finite matrices $(D^l)_l$ where each D^l has modes of order less or equal to l . We define

$$M_n^{s,\text{fin}} := \left\{ D^l = (d_{\alpha\beta}^l)_{\alpha\beta \in \mathbb{Z}_l^n} \mid d_{\alpha\beta}^l \in \mathbb{C} \ \& \ \exists C > 0 \ \forall l : |d_{\alpha\beta}^l| \leq \frac{C}{1+|\alpha-\beta|^s} \right\}. \quad (218)$$

Here for each $l \geq 1$ the set \mathbb{Z}_l^n denotes the set of modes of order less or equal to l .

It follows that for all t, k and

$$\left(\exp \left(D_b^{0,l} \frac{t}{k} \right) \exp \left(B_b^{0,l} \frac{t}{k} \right) \right)^k \in M_n^{s,\text{fin}} \quad (219)$$

for $s \geq n$, i.e., the finite approximations of the Trotter product formula are regular matrices indeed. Hence

$$\lim_{l \uparrow \infty} \lim_{k \uparrow \infty} \left(\exp \left(D_b^{0,l} \frac{t}{k} \right) \exp \left(B_b^{0,l} \frac{t}{k} \right) \right)^k \in M_n^s \quad (220)$$

Now consider finite approximations of the problem (157) to the cut-offs of this problem to a set of problems of finite modes with modes of order less or equal to l , i.e.,

$$\frac{d\mathbf{v}_l^{0,F}}{dt} = A_0^l \mathbf{v}_l^{0,F}. \quad (221)$$

For each l the solution

$$\mathbf{v}_l^{0,F} = \exp \left(A_0^l t \right) \mathbf{h}_l^F \quad (222)$$

is globally well-defined via the dissipative Trotter product formula. Our infinite linear algebra lemmas above imply that the sequence $\left(\mathbf{v}_l^{0,F} \right)_l$ is a Cauchy sequence for $s > n$ if $\mathbf{h}^F \in h^s(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$. Hence we have

Lemma 3.12. *We consider the torus of length $l = 1$ w.l.o.g.. Let $\mathbf{h}^F \in h^s(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$.*

$$\begin{aligned} \mathbf{v}_i^{0,F} &= \left(\exp \left(A_0 t \right) \mathbf{h}^F \right)_i \\ &= \lim_{l \uparrow \infty} \left(\lim_{k \uparrow \infty} \left(\exp \left(D_b^{0,l} \frac{t}{k} \right) \exp \left(B_b^{0,l} \frac{t}{k} \right) \right)^k \mathbf{h}_l^F \right)_i \in h^s(\mathbb{Z}^n) \end{aligned} \quad (223)$$

whenever $\mathbf{h}_i^F \in h^s(\mathbb{Z}^n)$ for $s > n$ for $1 \leq i \leq n$.

For the same reason

Corollary 3.13. *Consider the same situation as in the preceding lemma.*

$$\begin{aligned} \mathbf{v}_i^{r,0,F} &= \left(\exp \left(A_0^r t \right) \mathbf{h}^F \right)_i \\ &= \lim_{l \uparrow \infty} \left(\lim_{k \uparrow \infty} \left(\exp \left(D_b^{r,0,l} \frac{t}{k} \right) \exp \left(B_b^{r,0,l} \frac{t}{k} \right) \right)^k \mathbf{h}_l^F \right)_i \in h^s(\mathbb{Z}^n) \end{aligned} \quad (224)$$

whenever $\mathbf{h}_i^F \in h_l^s(\mathbb{Z}^n)$ for $s > n$ for $1 \leq i \leq n$.

Proof. Let $A_0^{r,l} = P_{M^l} A_0^r$ denote the projection of the matrix A_0^r to the finite matrix of modes less or equal to modes of order l . For finite vectors $\mathbf{v}_{i,l}^{r,0,F}$ with

$$\mathbf{v}_i^{r,0,F} = \lim_{l \uparrow \infty} \mathbf{v}_{i,l}^{r,0,F} \quad (225)$$

we have

$$\begin{aligned} \mathbf{v}_{i,l}^{r,0,F} &= \left(\exp \left(A_0^{r,l} t \right) \mathbf{h}^F \right)_i \\ &= \left(\lim_{k \uparrow \infty} \left(\exp \left(D_b^{r,0,l} \frac{t}{k} \right) \exp \left(B_b^{r,0,l} \frac{t}{k} \right) \right)^k \mathbf{h}_l^F \right)_i \end{aligned} \quad (226)$$

The right side of (226) is a Cauchy sequence with index l in M_n^s . \square

The stage $m = 0$ is special as the matrix $B_b^{r,0,l}$ does not depend on time. For this reason we get similar Trotter formulas for time derivatives using the damping of the dissipative factor $\exp \left(D_b^{r,0,l} \right)$. We shall use first order time derivatives formulas for linear problems with time-independent coefficients later when we approximate time dependent linear problems via time discretizations. We have

Corollary 3.14. *Recall that $A_0^{r,l} = P_{M^l} A_0^r$ denotes the projection of the matrix A_0^r to the finite matrix of modes less or equal to modes of order l .*

$$\begin{aligned} \frac{d}{dt} \mathbf{v}_i^{r,0,F} &= \frac{d}{dt} \left(\exp \left(A_0^r t \right) \mathbf{h}^F \right)_i \\ &= \lim_{l \uparrow \infty} \left(A_0^{r,l} \lim_{k \uparrow \infty} \left(\exp \left(D_b^{r,0,l} \frac{t}{k} \right) \exp \left(B_b^{r,0,l} \frac{t}{k} \right) \right)^k \mathbf{h}_l^F \right)_i \in h^s(\mathbb{Z}^n) \end{aligned} \quad (227)$$

whenever $\mathbf{h}_i^F \in h_l^s(\mathbb{Z}^n)$ for $s > n$ for $1 \leq i \leq n$.

Next at stage $m \geq 1$ we cannot apply the results above directly in order to define

$$(\mathbf{v}^{m,F}) = T \exp(A_m t) \mathbf{h}^F, \quad (228)$$

or in order to define

$$(\mathbf{v}^{r,m,F}) = T \exp(A_m^r t) \mathbf{h}^F. \quad (229)$$

The main difference of the stages $m \geq 1$ to the stage $m = 0$ is the time dependence of the coefficients formally expressed by the operators T (228) and (229) above. The second difference is that we need to control the zero modes if we want to establish a global scheme. For the latter reason, next we consider the controlled scheme (the considerations apply to the uncontrolled scheme as well for one iteration step). In each iteration step we construct the solution of an infinite linear ODE in dual space which corresponds to a linear partial integro-differential equation in the original space.

For each iteration step we need some subiterations in order to deal with the time-dependence of the coefficients. In order to apply our observations concerning linear algebra of infinite systems and the Trotter product formula, we consider time-discretizations. The time dependent formulas in (228) and (229) have rigorous definition via double limits (with respect to time and with respect to modes) of Trotter product formulas for finite systems. There are basically two possibilities to define a time-discretized scheme based on this form of a Trotter product formula. One possibility is to consider the successive linearized global problems at each stage m of the construction, where the matrices A_m^r are known in terms of the entries of the infinite vector $\mathbf{v}^{r,m-1,F}$ which contains information known from the previous step. According to the dissipative Trotter product formula we expect a time-discretization error of order $O(h)$ where h is an upper bound of the time step sizes. The other possibility is to apply the dissipative Trotter product formula locally and establish a time local limit $\lim_{m \uparrow \infty} \mathbf{v}^{r,m,F}$ at each time step. We shall consider the first possibility.

In any case, we define a time-discretized subscheme at each stage $m \geq 1$ and use the Trotter product formula for dissipative operators above locally in time in order to show that we get a converging subscheme. The convergence can be based on compactness arguments, and it can be based on global contraction with respect to a time-weighted norm introduced above. Another possibility is to establish a time-local contraction at each stage $m \geq 1$, i.e. in order to establish a solution $\mathbf{v}^{r,m,F}$ at stage m . The time-local contraction can be iterated due to the semi-group property of the operator such that the subscheme becomes global in time. At each iteration step m the time-discretized scheme for each global linear equation works for the uncontrolled scheme and the controlled scheme if we know that the data $\mathbf{v}^{m-1,F}$ (resp. $\mathbf{v}^{r,m-1,F}$) computed at the preceding step are bounded and regular, i.e., $\mathbf{v}^{r,m-1,F}(t) \in h^s(\mathbb{Z}^n)$ for $s \geq n+2$ with $n \geq 3$. Both subschemes have a limit with and without control function. It is in the limit with respect to the stage m that the control function r becomes useful. It also becomes useful for designing algorithms. Next we write down the subscheme for the uncontrolled subscheme and the controlled subschemes. Later we observe that the control function is globally bounded in order to prove that there is a limit as $m \uparrow \infty$. For each step m of the construction, however, the arguments for the controlled and the uncontrolled scheme are on the same footing. We describe the global linearized scheme, i.e., the scheme based on global linear equations which are equivalent to partial integro-differential equations in the original space. The procedure is defined recursively. For $t > 0$ arbitrary we define a scheme which converges on $[0, t]$ to the approximative solution $\mathbf{v}^{r,m,F}$. Having defined $\mathbf{v}^{m-1,F}$ at stage $m \geq 1$ we define a series $\mathbf{v}^{r,m,F,p}$, $p \geq 1$, where p is a natural number index which refers to a global time discretization t_q^p , $p \geq 1$ and where $q \in \{1, \dots, 2^p\}$ along with $t_q^p - t_{q-1}^p = 2^{-p}$ for all $p \geq 1$, and $t_0 := 0$, such that we get a contractive

scheme with respect to a certain time-weighted Sobolev norm. Next for given $p \geq 1$ and $q \in \{1, \dots, 2^p\}$ we define the equation for $\mathbf{v}^{r,m,F,p,q}$ on the interval $[t_{q-1}^p, t_q^p]$, and where the initial data are given by

$$\mathbf{v}^{r,m,F,p,q-1} \left(t_{q-1}^p \right). \quad (230)$$

Here for $q = 1$ we have the initial data

$$\mathbf{v}^{r,m,F,p,0} (t_0^p) = \mathbf{h}^{r,F}. \quad (231)$$

Note that the vector $\mathbf{h}^{r,F}$ equals the initial data \mathbf{h}^F , but without the zero modes, i.e.,

$$\mathbf{h}^{r,F} = \left(h_1^{r,F}, \dots, h_n^{r,F} \right)^T, \quad (232)$$

where for $1 \leq i \leq n$ we have

$$h_i^{r,F} = (h_{i\alpha})_{\alpha \in \mathbb{Z}^n \setminus \{0\}}. \quad (233)$$

Next we define for each $p \geq 1$ the sequence of local equations for $\mathbf{v}^{r,m,F,p,q}$. This leads to a global sequence $(\mathbf{v}^{r,m,F,p})_{p \geq 1}$ defined on $[0, T]$ for arbitrary $T > 0$. The next goal is to obtain a contraction result with respect to the iteration number p in order to establish the existence of global regular solutions $\mathbf{v}^{r,m,F}$ for the approximative problems

$$\frac{d\mathbf{v}^{r,m,F}}{dt} = A^{r,NS} (\mathbf{v}^{r,m-1}) \mathbf{v}^{r,m,F}, \quad (234)$$

where $\mathbf{v}^{r,m,F} = \left(\mathbf{v}_1^{r,m,F}, \dots, \mathbf{v}_n^{r,m,F} \right)^T$ and where for each $m \geq 1$ the initial data are given by $\mathbf{h}^{r,F}$. Here we look at $A^{r,NS}$ defined above as an operator such that $A^{r,NS} (\mathbf{v}^{r,m-1})$ is obtained by applying this operator to the argument $\mathbf{v}^{r,m-1}$ instead of \mathbf{v}^r . We shall give the details for the iterations steps $p, q \geq 1$. First we have to obtain $\mathbf{v}^{r,m,F,p,q}$ for each $p \geq 1$ and $q \in \{1, \dots, 2^p\}$. On the time interval $[t_{q-1}^p, t_q^p] = \left[\frac{q-1}{2^p} T, \frac{q}{2^p} T \right]$ we have the local Cauchy problem

$$\frac{d\mathbf{v}^{r,m,F,p,q}}{dt} = A^{r,NS} (\mathbf{v}^{r,m-1}) \mathbf{v}^{r,m,F,p,q}, \quad (235)$$

where $\mathbf{v}^{r,m,F,p,q} = \left(\mathbf{v}_1^{r,m,F,p,q}, \dots, \mathbf{v}_n^{r,m,F,p,q} \right)^T$ and where for each $m \geq 1$ the initial data are given by $\mathbf{v}^{r,m,F,p,q-1} \left(t_{q-1}^p \right)$. Next we explicitly describe the matrix $A^{r,NS} (\mathbf{v}^{r,m-1,p,q})$ which is a $n(\mathbb{Z}^n \setminus \{0\}) \times n(\mathbb{Z}^n \setminus \{0\})$ -matrix with

$$A^{r,NS} (\mathbf{v}^{r,m-1}) = \left(A_{ij}^{r,NS} (\mathbf{v}^{r,m-1}) \right)_{1 \leq i,j \leq n} \quad (236)$$

where for $1 \leq i, j \leq n$ the entry $A_{ij}^{r,NS}(\mathbf{v}^{r,m-1})$ is a $\mathbb{Z}^n \setminus \{0\} \times \mathbb{Z}^n \setminus \{0\}$ -matrix. We have

$$A^{r,NS}(\mathbf{v}^{r,m-1}) \mathbf{v}^{r,m,F,p,q} = \left(\sum_{j=1}^n A_{1j}^{r,NS}(\mathbf{v}^{r,m-1}) \mathbf{v}_1^{r,m,F,p,q}, \dots, \sum_{j=1}^n A_{nj}^{r,NS}(\mathbf{v}^{r,m-1}) \mathbf{v}_n^{r,m,F,p,q} \right)^T, \quad (237)$$

where for all $1 \leq i \leq n$

$$\sum_{j=1}^n A_{ij}^{r,NS}(\mathbf{v}^{r,m-1}) \mathbf{v}_j^{r,m,F,p,q} = \left(\left(\sum_{j=1}^n \sum_{\beta \in \mathbb{Z}^n} A_{i\alpha j\beta}^{r,NS}(\mathbf{v}^{r,m-1}) v_{j\beta}^{r,m,F,p,q} \right)_{\alpha \in \mathbb{Z}^n} \right)_{1 \leq i \leq n}^T. \quad (238)$$

The entries $A_{i\alpha j\beta}^{r,NS}(\mathbf{v}^{r,m-1})$ are determined by the equation as follows. On the diagonal, i.e., for $i = j$ and for $\alpha, \beta \neq 0$ we have the entries

$$\begin{aligned} \delta_{ij} A_{i\alpha j\beta}^{r,NS}(\mathbf{v}^{r,m-1}) &= \delta_{ij} \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) - \delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{l} v_{j(\alpha-\beta)}^{r,m-1} \\ &+ \delta_{ij} 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^{r,m-1}}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (239)$$

where for $\alpha = \beta$ the terms of the form $v_{k(\alpha-\beta)}^{r,m-1}$ are zero (such that we do not need to exclude these terms explicitly). Furthermore, off-diagonal we have for $i \neq j$ the entries

$$(1 - \delta_{ij}) A_{i\alpha j\beta}^{r,NS}(\mathbf{v}) = 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^{r,m-1}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \quad (240)$$

The idea for a global scheme is to determine $\mathbf{v}^{r,F} = \lim_{m \uparrow \infty} \mathbf{v}^{r,m,F}$ for a simple control function and a certain iteration

$$\frac{d\mathbf{v}^{r,m,F}}{dt} = A^{NS}(\mathbf{v}^{r,m-1}) \mathbf{v}^{r,m,F}, \quad (241)$$

starting with $\mathbf{v}^{r,0} := \mathbf{h}$. We shall use the abbreviation

$$A_m^r := A^{NS}(\mathbf{v}^{r,m-1}). \quad (242)$$

The proof of global regular existence can be obtained if a uniform upper bound for the sequence $(\mathbf{v}^{r,m,F})_{m \geq 1}$ can be found. An alternative is a contraction result. Contraction results have the advantage that they lead to uniqueness results with respect to the related Banach space. Next we define a Banach space in order to get a contraction result for the subscheme $(\mathbf{v}^{r,m,F,p})_{p \geq 1}$. Note that for each $p \geq 1$ the problem for $\mathbf{v}^{r,m,F,p}$ on $[0, T]$ is defined by 2^p recursively defined subproblems for $\mathbf{v}^{r,m,F,p,q}$ for $1 \leq q \leq 2^p$

which are defined on the interval $[t_{q-1}^p, t_q^p]$ where the data for the problem for $q \geq 2$ are defined by the final data of the subproblem for $\mathbf{v}^{r,m,F,p-q-1}$ evaluated at t_{q-1}^p .

Next consider the function space

$$B^{n,s} := \left\{ t \rightarrow \mathbf{u}^F(t) = (\mathbf{u}_1^F, \dots, \mathbf{u}_n^F)^T \mid \forall t \geq 0 \mathbf{u}_i^F(t) \in h^s(\mathbb{Z}^n \setminus \{0\}) \right\} \quad (243)$$

Note that we excluded the zero modes because this Banach space is designed for the controlled equations. For $\mathbf{u}^F \in B^{n,s}$ define

$$\left| \mathbf{u}^F \right|_{h^s, C}^{T, \exp} := \sup_{t \in [0, T]} \sum_{i=1}^n \exp(-Ct) \left| \mathbf{u}_i^F(t) \right|_{h^s}. \quad (244)$$

In the following we abbreviate

$$A_m^r = A^{NS}(\mathbf{v}^{r,m-1,F}). \quad (245)$$

Especially the evaluation of the right side of (245) at time t is denoted by $A_m^r(t)$. Recall that

$$\mathbb{Z}^{n,0} := \mathbb{Z}^n \setminus \{0\} \quad (246)$$

For $s > 0$ let us assume that

$$t \rightarrow \mathbf{v}^{r,m-1,F}(t) \in C^k([0, T], h^s(\mathbb{Z}^{n,0})) \quad (247)$$

Here $C^k([0, T], h^s(\mathbb{Z}^{n,0}))$ is the function space of $n \times (\mathbb{Z}^{n,0})$ vectors which have k -times differentiable component functions $t \rightarrow v_{i\alpha}^{r,m-1}(t)$ for $1 \leq i \leq n$ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Note that there is no need here to be very restrictive with respect to the degree of the Sobolev norm s . Each component function satisfies a Taylor formula. Hence we have the Taylor formula

$$\begin{aligned} \mathbf{v}^{r,m-1,F}(t+h) &= \sum_{0 \leq p \leq k-1} D_t^p \mathbf{v}^{r,m-1,F}(t) h^p \\ &+ \frac{h^p}{(k-1)!} \int_0^1 (1-\theta)^{k-1} D_t^k \mathbf{v}^{r,m-1,F}(t+\theta h) d\theta \end{aligned} \quad (248)$$

for $t \in [0, T-h]$ and $h > 0$. Here

$$D_t^p \mathbf{v}^{r,m-1,F}(t) := \left(\frac{d^p}{dt^p} v_{i\alpha}^{r,m-1} \right)_{1 \leq i \leq n, \alpha \in \mathbb{Z}^{n,0}}^T \quad (249)$$

In the following we assume that that $\mathbf{h}_i^{r,F}, \mathbf{v}_i^{r,m-1,F} \in h^s(\mathbb{Z}^{n,0})$ for $s = n+2$ and $1 \leq i \leq n$, because this property is inherited for $\mathbf{v}_i^{r,m,F}$ as we go from stage $m-1$ to stage m . Note that the matrix-valued function

$t \rightarrow A_m^r(t+h) - A_m^r(t) = A^{NS}(\mathbf{v}^{r,m-1,F}(t+h)) - A^{NS}(\mathbf{v}^{r,m-1,F}(t))$ is Lipschitz, i.e., for some finite Lipschitz constant $L > 0$ we have

$$\begin{aligned} & \left| (A^{NS}(\mathbf{v}^{r,m-1,F}(t+h)) - A^{NS}(\mathbf{v}^{r,m-1,F}(t))) \mathbf{h}^{r,F} \right|_{h^s} \\ & \leq L |\mathbf{v}^{r,m-1,F}(t+h) - \mathbf{v}^{r,m-1,F}(t)|_{h^s}. \end{aligned} \quad (250)$$

Using the infinite linear algebra lemmas above we observe

Lemma 3.15. *Let $s > n + 2$ and $T > 0$ be given. For $k = 2$ assume that $\mathbf{v}^{r,m-1,F}$ is regular as in (247). Then for some finite $C > 0$*

$$\sup_{u \in [0,T]} \exp(-Cu) \left| \mathbf{v}^{r,m,F,p}(u) - \mathbf{v}^{r,m,F,p-1}(u) \right|_{h^s} \leq \frac{L}{2^{p-1}}. \quad (251)$$

Proof. For notational reasons the size of the torus is assumed to be one. We remark that

$$t_{2(q-1)}^p = 2(q-1)2^{-p}T = (q-1)2^{-(p-1)}T = t_{q-1}^{p-1}, \quad (252)$$

and accordingly

$$A_m^r(t_{2(q-1)}^p) = A_m^r(t_{q-1}^{p-1}) \quad (253)$$

For some vector $\mathbf{C}_{q-1}^{h,p} \in h^s(\mathbb{Z}^n)$ for $s \geq n + 2$ we postulate the difference

$$\mathbf{v}^{r,m,F,p,q-1}(t_{2(q-1)}^p) - \mathbf{v}^{r,m,F,p-1,q-1}(t_{q-1}^{p-1}) = \mathbf{C}_{q-1}^{h,p}, \quad (254)$$

and we consider some properties which the vector $\mathbf{C}_q^{h,p}$ inherits from $\mathbf{C}_{q-1}^{h,p}$. Next at stage $p \geq 1$ consider the initial data $\mathbf{v}^{r,m,F}(t_{q-1}^{p-1})$ of the problem at substep $1 \leq q \leq 2^{p-1}$. We have $t \in [t_{2(q-1)}^p, t_{2q}^p] = [t_{q-1}^{p-1}, t_q^{p-1}]$, where it makes sense to consider the subintervals $[t_{2(q-1)}^p, t_{2q-1}^p]$ and $[t_{2q-1}^p, t_{2q}^p]$. We may consider $t \in [t_{2q-1}^p, t_{2q}^p]$ w.l.o.g. because the following estimate simplifies $t \in [t_{2q-2}^p, t_{2q-1}^p]$. For $t \in [t_{2q-1}^p, t_{2q}^p]$ we have

$$\begin{aligned} & \left| \mathbf{v}_i^{r,m,F,p,2q}(t) - \mathbf{v}_i^{r,m,F,p-1,q}(t) \right|_{h^s} \\ & \leq \left| \mathbf{v}_i^{r,m,F,p,2q}(t) - \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) - \left(\mathbf{v}_i^{r,m,F,p-1,q}(t) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right) \right|_{h^s} \\ & \quad + \left| \mathbf{v}_i^{r,m,F,p,2q-1}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right|_{h^s}. \end{aligned} \quad (255)$$

Since $t_{2(q-1)}^p = t_{q-1}^{p-1}$ and with (254) above for the last term in (256) we have

$$\begin{aligned}
& + \left| \mathbf{v}_i^{r,m,F,p,2q-1}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right|_{h^s} \\
& = \left| \left(\exp \left(A_m^r(t_{2(q-1)}^p) \left(t_{2q-1}^p - t_{2(q-1)}^p \right) \right) \mathbf{v}^{r,m,F,p,2q}(t_{2(q-1)}^p) \right)_i \right. \\
& \quad \left. - \left(\exp \left(A_m^r(t_{q-1}^{p-1}) \left(t_{2q-1}^p - t_{2(q-1)}^p \right) \right) \mathbf{v}^{r,m-1,F,p-1,q}(t_{q-1}^{p-1}) \right)_i \right|_{h^s} \quad (256) \\
& = \left| \exp \left(A_m^r(t_{2(q-1)}^p) \left(t_{2q-1}^p - t_{2(q-1)}^p \right) \right) \mathbf{C}_{q-1}^{h,p} \right|_{h^s} \\
& \leq \left| \exp \left(\frac{C}{4^p} \right) \mathbf{C}_{q-1}^{h,p} \right|_{h^s}
\end{aligned}$$

for some finite $C > 0$, which depends only on data known at stage $m-1$. Furthermore, for the first term on the right side of (256) we may use the rough estimate

$$\begin{aligned}
& \left| \mathbf{v}_i^{r,m,F,p,2q}(t) - \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) - \left(\mathbf{v}_i^{r,m,F,p-1,q}(t) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right) \right|_{h^s} \\
& \leq \left| \exp \left(A_m^r(t_{2q-1}^p) \left(t - t_{2q-1}^p \right) \right) \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) \right. \\
& \quad \left. - \left(\exp \left(A_m^r(t_{2q-1}^p) \left(t - t_{2q-1}^p \right) \right) \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right) \right|_{h^s} \quad (257)
\end{aligned}$$

The right side of (257) we observe with (256)

$$\begin{aligned}
& \left| \exp \left(A_m^r(t_{2q-1}^p) \left(t - t_{2q-1}^p \right) \right) \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right. \\
& \quad \left. + \mathbf{v}_i^{r,m,F,p,2q}(t_{2q-1}^p) - \mathbf{v}_i^{r,m,F,p-1,q}(t_{2q-1}^p) \right|_{h^s} \\
& \leq 2 \left| \exp \left(\frac{2C}{4^p} \right) \mathbf{C}_{q-1}^{h,p} \right|_{h^s} \quad (258)
\end{aligned}$$

As for each $p \geq 1$ the entries of the sequence $\mathbf{C}_q^{h,p}$ are in $O(h^2)$ where h denotes the maximal time step size (which is 2^{-p} with our choice) the difference to be estimated is in $O(h)$ and we are done. \square

The preceding lemma shows that we have a Cauchy sequence

$$(\mathbf{v}^{r,m,F,p})_{p \geq 1} \quad (259)$$

with respect to a regular (time weighted) norm and with a limit $\mathbf{v}^{r,m,F}$ with

$$\mathbf{v}_i^{r,m,F}(t) \in h^s(\mathbb{Z}^{n,0}) \quad (260)$$

for all $t \in [0, T]$. Since we have a dissipative term (damping exponential) in the Trotter product formula, similar observations can be made for the time derivative sequence

$$\left(\frac{d}{dt} \mathbf{v}^{r,m,F,p} \right)_{p \geq 1}. \quad (261)$$

Note, however, that $s \geq n + 2 \geq 5$ above may be arbitrary, and this can be exploited in order to prove regularity with respect to time t for each $\mathbf{v}^{r,m,F}$ via the defining equation of the latter function. We even do not need estimates for products of functions in Sobolev spaces which may be borrowed from classical Sobolev space analysis. Using the lemma above for large $s > 0$ we may instead use the regularity implied by infinite matrix products as pointed out above. As a consequence of the preceding lemma we note

Lemma 3.16. *For all $m \geq 1$ and $s > n + 2 \geq 5$ the function*

$$(\mathbf{v}^{r,m,F})_i = (T \exp(A_m^r t) \mathbf{h}^{r,F})_i \in h_l^s(\mathbb{Z}^{n,0}), \quad (262)$$

is well-defined, whenever $\mathbf{h}_i^F \in h_l^s(\mathbb{Z}^n)$.

The same holds for the uncontrolled approximations, of course. We note

Corollary 3.17. *For all $m \geq 1$ and $s > n + 2$ the function*

$$(\mathbf{v}^{m,F})_i = (T \exp(A_m t) \mathbf{h}^F)_i \in h_l^s(\mathbb{Z}^n) \quad (263)$$

is well-defined, whenever $\mathbf{h}_i^F \in h_l^s(\mathbb{Z}^n)$.

However, it is essential to get a uniformly bounded sequence

$$(\mathbf{v}^{r,m,F})_{m \in \mathbb{N}} = (T \exp(A_m^r t) \mathbf{h}_i^F \in h_l^s(\mathbb{Z}^n))_{m \in \mathbb{N}}. \quad (264)$$

for some $\nu > 0$. We remarked in the introduction that we can even choose $\nu > 0$. Indeed, we observed that we can choose it arbitrarily large (as is also well-known). However this was not needed so far and we shall not need it later on. It is just an useful observation in order check algorithms in the most simple situation via equivalent formulations with rigorous damping.

At this point it is useful to consider the controlled sequence $(\mathbf{v}_i^{r,m,F})_{m \in \mathbb{N}, 1 \leq i \leq n}$.

Recall that the functions $\mathbf{v}^{r,m,F}$ are designed such that the zero modes are zero $v_{i0}^{r,m} = 0$. The control function is just defined this way. Next we show that a uniformly bounded controlled sequence $(\mathbf{v}_i^{r,m,F})_{m \in \mathbb{N}, 1 \leq i \leq n}$ implies uniformly boundedness of the uncontrolled sequence $(\mathbf{v}_i^{m,F})_{m \in \mathbb{N}, 1 \leq i \leq n}$. In order to observe this we go back to (21). At stage $m \geq 1$ it is assumed the $v_{i0}^{r,m-1} = 0$. The controlled approximating equation at stage m is obtained

from (21) by elimination of the zero modes. We have for $1 \leq i \leq n$ and $\alpha \neq 0$ the equation

$$\begin{aligned} \frac{dv_{i\alpha}^{r,m}}{dt} = & \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^{r,m} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{r,m-1} v_{i\gamma}^{r,m} \\ & + 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{r,m-1} v_{k(\alpha-\gamma)}^{r,m}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (265)$$

We considered the controlled equation systems as autonomous systems of non-zero modes. However, in order to compare the controlled system with the original one we may define

$$\frac{dv_{i0}^{r,m}}{dt} = - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(-\gamma)}^{r,m-1} v_{i\gamma}^{r,m}. \quad (266)$$

Note that $\mathbf{v}_i^{r,m-1,F}, \mathbf{v}_i^{r,m,F} \in h^s(\mathbb{Z}^{n,0})$ for $s > n \geq 3$ implies that the right side of (266) is bounded by constant finite $C > 0$. Hence we have

Lemma 3.18. *If the sequence $(\mathbf{v}^{r,m,F})_{m \geq 1}$ has an upper bound $c > 0$ with respect to the $|\cdot|_s = \sum_{i=1}^n |\cdot|_{h^s}$ -norm, we have for some finite $C > 0$*

$$|r_0(t)| \leq \exp(Ct) \quad (267)$$

for all $t \geq 0$.

It remains to show that for some finite $C > 0$ such that for all $1 \leq i \leq n$ and $m \geq 0$ we have

$$|\mathbf{v}_i^{r,m,F}(t, \cdot)|_{h_i^s} \leq C \quad (268)$$

Based on the arguments so far there are several ways to get an uniform bound for the sequence $(\mathbf{v}^{r,m,F})_{m \geq 1}$. One is via contraction results on certain balls in appropriate function spaces. The radius of such a ball clearly depends on the size of the initial data and on the size of the horizon. However it is sufficient that for each dual Sobolev norm index $s > 0$ and for each data size $|\mathbf{h}^F|_s$ and each horizon size $T > 0$ we find a contraction result on an appropriate ball for a related time weighted function space. Let's look at the details. For arbitrary $T > 0$ consider two smooth vector-valued functions on the n -torus, i.e., functions of the form

$$\mathbf{f}, \mathbf{g} \in [C^\infty([0, T], \mathbb{T}^n)]^n. \quad (269)$$

Consider the equations

$$\frac{d\mathbf{v}^{r,f,F}}{dt} = A^{r,NS}(\mathbf{f}) \mathbf{v}^{r,f,F}, \quad (270)$$

along with $\mathbf{v}^{r,f,F}(0) = \mathbf{h}^{r,F}$, and

$$\frac{d\mathbf{v}^{r,g,F}}{dt} = A^{r,NS}(\mathbf{g}) \mathbf{v}^{r,g,F}, \quad (271)$$

along with $\mathbf{v}^{r,g,F}(0) = \mathbf{h}^{r,F}$. Here we denote $\mathbf{v}^{r,f,F} = (\mathbf{v}_1^{r,f,F}, \dots, \mathbf{v}_n^{r,f,F})^T$ and similarly for the function $\mathbf{v}^{r,g,F}$. As in or notation above the matrix $A^{r,NS}(\mathbf{f})$ is a $n\mathbb{Z}^n \times n\mathbb{Z}^n$ -matrix with

$$A^{r,NS}(\mathbf{f}) = \left(A_{ij}^{r,NS}(\mathbf{f}) \right)_{1 \leq i,j \leq n} \quad (272)$$

where for $1 \leq i, j \leq n$ the entry $A_{ij}^{r,NS}(\mathbf{f})$ is a $\mathbb{Z}^n \times \mathbb{Z}^n$ -matrix. We define

$$A^{r,NS}(\mathbf{f}) \mathbf{v}^{r,f,F} = \left(\sum_{j=1}^n A_{1j}^{r,NS}(\mathbf{f}) \mathbf{v}_1^{r,f,F}, \dots, \sum_{j=1}^n A_{nj}^{r,NS}(\mathbf{f}) \mathbf{v}_n^{r,f,F} \right)^T, \quad (273)$$

where for all $1 \leq i \leq n$

$$\sum_{j=1}^n A_{ij}^{r,NS}(\mathbf{f}) \mathbf{v}_j^{r,f,F} = \left(\left(\sum_{j=1}^n \sum_{\beta \in \mathbb{Z}^n} A_{i\alpha j \beta}^{r,NS}(\mathbf{f}) v_{j\beta}^{r,f,F} \right)_{\alpha \in \mathbb{Z}^n} \right)_{1 \leq i \leq n}^T. \quad (274)$$

The entries $A_{i\alpha j \beta}^{r,NS}(\mathbf{f})$ of $A^{r,NS}(\mathbf{v})$ are determined as follows. On the diagonal, i.e., for $i = j$ we have the entries for $\alpha, \beta \neq 0$

$$\begin{aligned} \delta_{ij} A_{i\alpha j \beta}^{r,NS}(\mathbf{f}) &= \delta_{ij} \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) - \delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{l} f_{j(\alpha-\beta)} \\ &+ \delta_{ij} 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) f_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi \alpha_i^2}, \end{aligned} \quad (275)$$

where for $\alpha = \beta$ the terms of the form $f_{k(\alpha-\beta)}$ are zero (such that we do not need to exclude these terms explicitly). Furthermore, off-diagonal we have for $i \neq j$ the entries

$$(1 - \delta_{ij}) A_{i\alpha j \beta}^{r,NS}(\mathbf{f}) = 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) f_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \quad (276)$$

The definition of $A^{r,NS}(\mathbf{g})$ is analogous. Next for functions $\mathbf{u}^{r,F} = (\mathbf{u}_1^{r,F}, \dots, \mathbf{u}_n^{r,F})$ and for $s \geq n+2$ consider the norm

$$|\mathbf{u}^{r,F}|_s := \sum_{i=1}^n |\mathbf{u}_i^{r,F}|_{h^s}. \quad (277)$$

Consider a ball of radius $2|\mathbf{h}^{r,F}|_s$ around the origin, i.e., consider the ball

$$B_{2|\mathbf{h}^{r,F}|_s} := \{ \mathbf{u}^{r,F} \mid |\mathbf{u}^{r,F}|_s \leq 2|\mathbf{h}^{r,F}|_s \}. \quad (278)$$

For $s \geq n+2$ and for data $\mathbf{h}^{r,F} \in h^s(\mathbb{Z}^n \setminus \{0\})$ the considerations above shows that the Cauchy problem

$$\frac{d\mathbf{v}^{r,f,F}}{dt} = A^{r,NS}(\mathbf{f}) \mathbf{v}^{r,f,F}, \quad (279)$$

along with $\mathbf{v}^{r,f,F}(0) = \mathbf{h}^{r,F}$ has a regular solution $\mathbf{v}^{r,f,F}$ with $\mathbf{v}^{r,f,F}(t) \in h^s(\mathbb{Z}^n \setminus \{0\})$. Moreprecisely, for $s = n + 2 \geq 5$ we have

$$|\mathbf{v}^{r,f,F}|_{h^s}^T \leq 2|\mathbf{h}^{r,F}|_s T. \quad (280)$$

Next for linear operator

$$(\mathbf{f} - \mathbf{g}) \rightarrow \left(A_{i\alpha j\beta}^{r,NS}(\mathbf{f}) \right) \mathbf{u}^{r,F} - \left(A_{i\alpha j\beta}^{r,NS}(\mathbf{g}) \right) \mathbf{u}^{r,F} \quad (281)$$

are Lipschitz with some Lipschitz constant L on this ball. Note that we have

$$\begin{aligned} & A_{i\alpha j\beta}^{r,NS}(\mathbf{f}) - A_{i\alpha j\beta}^{r,NS}(\mathbf{g}) \\ &= -\delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{t} (f_{j(\alpha-\beta)} - g_{j(\alpha-\beta)}) \\ &+ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) (f_{k(\alpha-\beta)} - g_{k(\alpha-\beta)})}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (282)$$

Furthermore, for $\mathbf{g}^{r,F} \in B_{2|\mathbf{h}^{r,F}|_s}$ we may assume w.l.o.g. that the Lipschitz constant L is chosen such that

$$\sup_{\mathbf{g}^{r,F} \in B_{2|\mathbf{h}^{r,F}|_s}} \left| \left(\delta_{ij} A_{i\alpha j\beta}^{r,NS}(\mathbf{g}) \right) \mathbf{u}^{r,F} \right|_{h^s} \leq L |\mathbf{u}^{r,F}|_{h^{s-2}} \quad (283)$$

where the weaker norm on the right side of (283) is due to the fact that we have to compensate the first term on the right side of

$$\begin{aligned} & \delta_{ij} A_{i\alpha j\beta}^{r,NS}(\mathbf{g}) = \delta_{ij} \sum_{j=1}^n \nu \left(-\frac{4\pi \alpha_j^2}{t^2} \right) - \delta_{ij} \sum_{j=1}^n \frac{2\pi i \beta_j}{t} g_{j(\alpha-\beta)} \\ &+ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) g_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (284)$$

Lemma 3.19. *Let $T > 0$ be arbitrary, and let $\mathbf{f}^{r,F}, \mathbf{g}^{r,F} \in B_{2|\mathbf{h}^{r,F}|_s}$ for $s \geq n + 3$. For $C \geq 3l$ we have*

$$\left| \mathbf{v}^{r,f,F} - \mathbf{v}^{r,g,F} \right|_{h^s,C}^{T,exp} \leq \frac{1}{2} \left| \mathbf{f}^F - \mathbf{g}^F \right|_{h^s,C}^{T,exp} \quad (285)$$

For $C \geq 6L$ we have

$$\left| \mathbf{v}^{r,f,F} - \mathbf{v}^{r,g,F} \right|_{h^s,C}^{T,exp,1} \leq \frac{1}{2} \left| \mathbf{f}^F - \mathbf{g}^F \right|_{h^s,C}^{T,exp,1} \quad (286)$$

Proof. For each $t \in [0, T]$ we have

$$\begin{aligned}
& \left| \mathbf{v}^{r,f,F}(t) - \mathbf{v}^{r,g,F}(t) \right|_{h^s} \\
& \leq \left| \int_0^t A^{r,NS}(\mathbf{f})(u) \mathbf{v}^{r,f,F}(u) du - \int_0^t A^{r,NS}(\mathbf{g})(u) \mathbf{v}^{r,g,F}(u) du \right|_{h^s} \\
& \leq \left| \left(\int_0^t A^{r,NS}(\mathbf{f})(u) - \int_0^t A^{r,NS}(\mathbf{g})(u) \right) \mathbf{v}^{r,f,F}(u) du \right|_{h^s} \\
& \quad + \left| \int_0^t A^{r,NS}(\mathbf{g})(u) (\mathbf{v}^{r,f,F}(u) - \mathbf{v}^{r,g,F}(u)) du \right|_{h^s} \\
& \leq LT \sup_{u \in [0, T]} \left| \mathbf{f}^F(u) - \mathbf{g}^F(u) \right|_{h^s} \\
& \quad + LT \sup_{u \in [0, T]} \left| \mathbf{v}^{r,f,F}(u) - \mathbf{v}^{r,g,F}(u) \right|_{h^{s-2}}.
\end{aligned} \tag{287}$$

It follows that

$$\begin{aligned}
& \left| \mathbf{v}^{r,f,F}(t) - \mathbf{v}^{r,g,F}(t) \right|_{h^s} \\
& \leq L \left| \mathbf{f} - \mathbf{g} \right|_{h^s, C}^{T, \exp} \int_0^t \exp(Cu) du \\
& \quad + L \left| \mathbf{v}^{r,f,F} - \mathbf{v}^{r,g,F} \right|_{h^s, C}^{T, \exp} \int_0^t \exp(Cu) du \\
& \leq \frac{L \exp(Ct)}{C} \left| \mathbf{f}^F - \mathbf{g}^F \right|_{h^s, C}^{T, \exp} \\
& \quad + \frac{L \exp(Ct)}{C} \sup_{u \geq 0} \left| \mathbf{v}^{r,f,F}(u) - \mathbf{v}^{r,g,F}(u) \right|_{h^{s-2}, C}^{T, \exp}
\end{aligned} \tag{288}$$

Since

$$\left| \mathbf{v}^{r,f,F} - \mathbf{v}^{r,g,F} \right|_{h^{s-2}, C}^{T, \exp} \leq \left| \mathbf{v}^{r,f,F} - \mathbf{v}^{r,g,F} \right|_{h^s, C}^{T, \exp} \tag{289}$$

it follows that

$$\begin{aligned}
& \left| \mathbf{v}^{r,f,F}(\cdot) - \mathbf{v}^{r,g,F}(\cdot) \right|_{h^s, C}^{T, \exp} \\
& \leq \left(\frac{1}{(1-\frac{L}{C})} \frac{L}{C} \right) \left| \mathbf{f}^F(u) - \mathbf{g}^F(u) \right|_{h^s, C}^{T, \exp}.
\end{aligned} \tag{290}$$

For $C = 3L$ the result follows. The reasoning for the stronger norm is similar. \square

It is clear that this contraction result leads to global existence: for each $s \geq n + 2$ and $T > 0$ we find a constant $C > 0$ such that

$$\left| \mathbf{v}^{r,m,F} \right|_{h^s, C}^{T, \exp} \leq C \quad (291)$$

Uniqueness follows by standard arguments. Uniform upper bounds for the approximative (controlled) solutions $\mathbf{v}^{r,m,F}$ lead to existence via compactness as well. Note that the infinite vectors $\mathbf{v}_i^{r,m,F}(t) = (v_{i\alpha}^{r,m}(t))_{\alpha \in \mathbb{Z}^{n,0}}$ are in 1-1 correspondence with classical functions

$$v_i^{r,m}(t, x) = \sum_{\alpha \in \mathbb{Z}^{n,0}} v_{i\alpha}^{r,m} \exp\left(\frac{2\pi i \alpha x}{l}\right), \quad (292)$$

where $v_i^{r,m} \in H^s(\mathbb{T}^n)$ for $s > n + 2$. Recall that

Theorem 3.20. *For $r > s$ and for any compact Riemann manifold M (and especially for $M = \mathbb{T}_l^n$) we have a compact embedding*

$$e : H^r(M) \rightarrow H^s(M) \quad (293)$$

This means that $(v_i^{r,m})_{m \in \mathbb{N}}$ has a convergent subsequence in $H^r(\mathbb{T}_l^n)$ for $r > s$ which corresponds to converging subsequence in the corresponding Sobolev space of infinite vectors of modes. Hence, passing to an appropriate subsequence $\mathbf{v}_i^{r,m',F}(t)$ of $\mathbf{v}_i^{r,m,F}(t)$ we have a limit

$$\mathbf{v}_i^{r,F}(t) = \lim_{m' \uparrow \infty} \mathbf{v}_i^{r,m',F}(t) \in h^r(\mathbb{Z}^n) \quad (294)$$

for $r < s$ (Rellich embedding). Since s is arbitrary this limit exists in $h^r(\mathbb{Z}^n)$ for all $r \in \mathbb{R}$. Hence, for all $1 \leq i \leq n$ we have a family $(\mathbf{v}_i^{r,m',F})_{m' \in \mathbb{N}}$ which satisfies

$$\begin{aligned} \frac{dv_{i\alpha}^{r,m'}}{dt} &= \sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^{r,m'} - \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{r,m'-1} v_{i\gamma}^{r,m'} \\ &+ 2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{0, \alpha\}} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{r,m'-1} v_{k(\alpha-\gamma)}^{r,m'}}{\sum_{i=1}^n 4\pi \alpha_i^2}. \end{aligned} \quad (295)$$

If we can prove that the limit of infinite vectors $\left(\frac{dv_{i\alpha}^{r,m'}}{dt}(t) \right)_{\alpha \in \mathbb{Z}^n}$ is continuous in the sense that

$$\begin{aligned} \lim_{m' \uparrow \infty} \left(\frac{dv_{i\alpha}^{r,m'}}{dt}(t) \right)_{\alpha \in \mathbb{Z}^n} &\in C(\mathbb{Z}^n) \\ &:= \left\{ (g_\alpha) \mid \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \exp\left(\frac{2\pi i \alpha x}{l}\right) \in C(\mathbb{Z}^n) \right\}, \end{aligned} \quad (296)$$

then we obtain a classical solution. Inspecting the terms on the right side the assumption that $s > 2 + n$ and $n \geq 2$ is more than sufficient in order to get

$$\left(\sum_{j=1}^n \nu \left(-\frac{4\pi\alpha_j^2}{l^2} \right) v_{i\alpha}^{r,m'} \right)_{\alpha \in \mathbb{Z}^n} \in h^{s-2}(\mathbb{Z}^n) \subset h^n(\mathbb{Z}^n), \quad (297)$$

such that with this assumption we may ensure that $\left(\frac{dv_{i\alpha}^{r,m'}}{dt} \right)_{m \in \mathbb{N}}(t)$ converges in $C(\mathbb{Z}^n) \subset h^r(\mathbb{Z}^n)$ with $r > \frac{1}{2}n$ if we can control the expressions for the convection term and for the Leray projection term appropriately. However, this is easily done with the help of the infinite linear algebra results above. We stick to $s > n + 2$ and $n \geq 2$. First for the convection term for each $1 \leq i \leq n$ and all $\alpha \in \mathbb{Z}^n$ we consider

$$- \sum_{j=1}^n \sum_{\gamma \in \mathbb{Z}^n \setminus \{\alpha\}} \frac{2\pi i \gamma_j}{l} v_{j(\alpha-\gamma)}^{r,m'-1} v_{i\gamma}^{r,m'}. \quad (298)$$

We observe $|\gamma_j v_{i\gamma}^{r,m'}|_{h^{s-1}(\mathbb{Z}^n)} \leq C$ for some constant $C > 0$ independent of m , hence

$$\left| \left(\sum_{\gamma \in \mathbb{Z}^n} v_{j(\alpha-\gamma)}^{r,m'-1} \gamma_j v_{i\gamma}^{r,m'} \right) \right|_{\alpha \in \mathbb{Z}^n} \in h^{2s-1-n} \leq C \quad (299)$$

for some $C > 0$ independent of m such that the limit is in $h^2(\mathbb{Z}^n)$. Hence (298) and the limit for $m' \uparrow \infty$ is in $h^2(\mathbb{Z}^n)$. Similarly, the Leray projection term

$$2\pi i \alpha_i 1_{\{\alpha \neq 0\}} \frac{\sum_{j,k=1}^n \sum_{\gamma \in \mathbb{Z}^n} 4\pi \gamma_j (\alpha_k - \gamma_k) v_{j\gamma}^{r,m'-1} v_{k(\alpha-\gamma)}^{r,m'}}{\sum_{i=1}^n 4\pi \alpha_i^2} \quad (300)$$

is bounded by a product of two infinite vectors which have some uniform bound $C > 0$ in $h^{s-1}(\mathbb{Z}^n)$, such that the Leray projection term is safely in $h^{2(s-1)-n}(\mathbb{Z}^n) \subset h^2(\mathbb{Z}^n)$ where we did not even take the $|\alpha|^2$ in the denominator corresponding to the Laplacian kernel into account. We have shown

Lemma 3.21. *For $s > n + 2$ and $n \geq 2$, and for the same $\nu > 0$ as above there is a $C > 0$ such that for all $1 \leq i \leq n$ and $m' \geq 0$ we have*

$$\left| \frac{d}{dt} \mathbf{v}_i^{r,m',F}(t) \right|_{h_i^s} \leq C \quad (301)$$

uniformly for $t > 0$, and

$$\frac{d}{dt} \mathbf{v}_i^{r,F}(t) = \lim_{m' \uparrow \infty} \frac{d}{dt} \mathbf{v}_i^{r,m',F}(t) \in h^r(\mathbb{Z}^n) \subset C(\mathbb{Z}^n) \subset h^2(\mathbb{Z}^n). \quad (302)$$

We conclude

Theorem 3.22. *The function*

$$\mathbf{v}_i^{r,F}(t) = \lim_{m' \uparrow \infty} \mathbf{v}_i^{r,m',F}(t), \quad 1 \leq i \leq n \quad (303)$$

satisfies the infinite nonlinear ODE equivalent to the controlled incompressible Navier-Stokes equation on the n -torus in a classical sense. Moreover, since the argument above can be repeated with arbitrary large $s > 0$ we have that for all $1 \leq i \leq n$ the infinite vector $\mathbf{v}_i^{r,F}(t)$ and its time derivative are in $h^s(\mathbb{Z}^n)$. Higher order time derivatives also exist in a classical sense by an analogous argument for derivatives of the Navier-Stokes equation.

Finally we have

Theorem 3.23. *Let $h_i \in C^\infty(\mathbb{T}^n)$. For each $\nu > 0$ and $l > 0$ and for all $1 \leq i \leq n$ and all $t \geq 0$*

$$v_i(t, \cdot) \in C^\infty(\mathbb{T}_l^n). \quad (304)$$

and

$$\mathbf{v}_i^F(t) \in h^s(\mathbb{T}_l^n). \quad (305)$$

for arbitrary $s \in \mathbb{R}$.

Proof. We have $v_i^F(t) = v_i^{r,F}(t) - r(t) \in h^s(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$, because $v_i^{r,F}(t) \in h^s(\mathbb{Z}^n)$ for all $s \in \mathbb{R}$, and $r(t)$ is a constant. The second part follows from Corollary above. Given $s > 0$ we can differentiate

$$v_i(t, \cdot) = \sum_{\gamma \in \mathbb{Z}^n} v_{i\gamma} \exp\left(\frac{2\pi i \gamma x}{l}\right), \quad (306)$$

up to order m , where m is the largest integer less than s , and get a Fourier series which converges in $L^2(\mathbb{T}_l^n)$. Hence,

$$v_i(t, \cdot) \in H^s(\mathbb{T}_l^n), \quad (307)$$

for all $t \geq 0$. Since, this is true for all for all $s > 0$ we the first statement of this lemma is true. \square

Now for a fixed $l > 0$ and any $\mathbf{h}_1^F \in h^s(\mathbb{Z}^n)$ for $s \geq pn + 2$ we have

$$\frac{d^p}{dt^p} \mathbf{v}_i^{m,F}(t) \in h_l^{s-pn}(\mathbb{Z}^n) \quad (308)$$

for all $m \geq 0$ from the uniform bound

$$|\mathbf{v}_i^{m,F}(t)|_{h_l^s} \leq C. \quad (309)$$

4 A converging algorithm

The infinite ODE-system can be approximated by finite ODE systems of with modes of order less or equal to some positive integer k in the sense of a projection on the modes of order $|\alpha| \leq k$. At a fixed time $t \geq 0$ this is a projection from the intersection $\cap_{s \in \mathbb{R}} h^s(\mathbb{Z}^n)$ of dual Sobolev spaces into the space of (even globally) analytic functions (finite Fourier series). The Trotter product formula for dissipative operators considered above holds for the system of finite modes, of course. More importantly, the resulting schemes based on this Trotter product formula approximate the corresponding scheme of infinite modes if the limit $k \uparrow \infty$ is considered. In this section we define this approximating scheme of finite modes. It is not difficult to show that the error of this scheme converges to zero as $k \uparrow \infty$. Detailed error estimates which relate the maximal mode of the finite system and dimension n and viscosity ν to the error in h^s norm are of interest in order to design algorithms. This deserves a closer investigation which will be done elsewhere. In this section we define algorithms of different approximation order in time via finite-mode approximations of the non-linear infinite ODEs which are equivalent to the incompressible Navier-Stokes equation. Furthermore some observations for the choice of parameters of the size of the domain $l > 0$ and the viscosity $\nu > 0$ are made, and we observe reductions to a cosine basis (symmetric data). We consider the controlled scheme which solves an equivalent equation for the sake of simplicity. From our description in the introduction it is clear how to compute an approximate solution of the incompressible Navier-Stokes equation from these data. Recall the representation of the controlled Navier-Stokes equation in terms of infinite matrices. It makes sense to formulate an 'algorithm' first for the infinite nonlinear ODE system. Accordingly, in the following the use of the word 'compute' related to substeps in an infinite scheme is not meant in a strict sense. It may be defined in some more strict sense by use of transfinite Turing machines, but what we have in mind is the finite approximations of an infinite object such that schemes can be implemented eventually. This is not an algorithm in a strict sense (even the description is not finite), but the projection to a set of finite modes leads to the algorithm immediately once it is described in the infinite set-up. Consider the limit $\mathbf{v}^{r,F} = (v_{i\alpha}^r)_{\alpha \in \mathbb{Z}^n}$ of the construction of the last section, i.e., the global solution of the controlled infinite controlled ODE system, as given. Then we may write this function formally in terms of the Dyson formalism as

$$v_i^{r,F}(t) = T \exp(A^r t) \mathbf{h}_i^F \in h_i^s(\mathbb{Z}^n), \quad (310)$$

where

$$\begin{aligned} T \exp(A^r t) &:= \sum_{m=0}^{\infty} \frac{1}{m!} \int_{[0,t]} dt_1 \cdots dt_k T A^r(t_1) \cdots A^r(t_k) dt_1 \cdots dt_k \\ &:= \sum_{k=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{k-1}} A^r(t_1) \cdots A^r(t_k), \end{aligned} \quad (311)$$

where

$$A^r(t_j) := (A^{r,ij})_{1 \leq i,j \leq n} \quad (312)$$

and for all $1 \leq i, j \leq n$ we have

$$A^{r,ij} = (a_{\alpha\beta}^{r,ij})_{\alpha,\beta \in \mathbb{Z}^n}, \quad (313)$$

where

$$a_{\alpha\beta}^{r,ij} = \delta_{ij} \left(\delta_{\alpha\beta} \nu \left(- \sum_{j=1}^n \frac{4\pi\alpha_j^2}{l^2} \right) - \sum_{j=1}^n \frac{2\pi i \beta_j v_{j(\alpha-\beta)}}{l} \right) + L_{ij}^v, \quad (314)$$

along with

$$L_{ij}^v = 2\pi i \alpha_i \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \quad (315)$$

Note that the modes $v_{i\alpha}$ are time-dependent - for this reason the Dyson time-order operator appears in this formal representation. In order to construct a computable approximation of this formula we need a stable dissipation term for the second order part of the operator. We can achieve this if we may use the Trotter product formula above. because the factor $\exp((\delta_{ij} D^r))$ is indeed in the regular matrix space M_n^s and it is a damping factor for the controlled system. In order to apply the Trotter product formula to the controlled Navier-Stokes system we may consider a time discretization and apply this formal time step by time step. Starting with $v_i^{r,t_0} := h_i$ let us assume that we have defined a scheme for $0 = T_0 < t_1 < \dots < t_k$ and computed the modes $v_{i,\alpha}^{r,t_l}$ for $0 \leq l \leq t_k$. Then in order to define the scheme recursively on the interval $[t_k, t_{k+1}]$ we first consider the natural splitting of the operator and define

$$A^r(t_j) = (\delta_{ij} D) + (B^{r,ij}(t_k)), \quad (316)$$

where $B^{r,ij}(t_k) = (b_{\alpha\beta}^{r,t_k,ij})$ along with

$$b_{\alpha\beta}^{r,t_k,ij} = \left(- \sum_{j=1}^n \frac{2\pi i \beta_j v_{j(\alpha-\beta)}^{r,t_k}}{l} \right) + L_{ij}^{t_k}, \quad (317)$$

and

$$L_{ij}^{t_k} = 2\pi i \alpha_i \frac{\sum_{k=1}^n 4\pi \beta_j (\alpha_k - \beta_k) v_{k(\alpha-\beta)}^{r,t_k}}{\sum_{i=1}^n 4\pi^2 \alpha_i^2}. \quad (318)$$

The most simple infinite scheme we could have in mind is then a scheme approximating $\mathbf{v}^{r,F}(t) = (v_{i\alpha}^r(t))_{\alpha \in \mathbb{Z}^n}$ for arbitrary given $t > 0$ in m steps then is

$$\begin{aligned} & \left(\exp \left(\frac{t}{m} (\delta_{ij} D) \right) \exp \left(\frac{t}{k} (B^{r,ij}(t_m)) \right) \right) \times \\ & \dots \left(\exp \left(\frac{t}{m} (\delta_{ij} D) \right) \exp \left(\frac{t}{k} (B^{r,ij}(t_0)) \right) \right) \mathbf{h}. \end{aligned} \quad (319)$$

This is indeed a time-dependent Trotter-product type approximation of order $o\left(\frac{1}{m}\right) \downarrow 0$ as $m \uparrow \infty$, where o denotes the small Landau o . Higher order approximations can be achieved taking account of some correction terms in the Trotter product formula. Indeed, consider 'truncations of order q ' of the term defined in (196) above. For each t_r , $0 \leq r \leq m$ consider

$$\begin{aligned} C^q(t_r) &= (\delta_{ij}D) + (B^{r,ij}(t_r)) + \\ &\sum_{p=1}^q \frac{1}{p!} \sum_{l \geq 1, \beta^l \in \mathbb{N}_{10}^l, m' + |\beta^l| = p, l \leq m'+1} (\delta_{ij}D)^{m'} \times \\ &\times I_{\beta^{m'}} [\Delta(B^{r,ij}(t_r)), (B^{r,ij}(t_r))]_T. \end{aligned} \quad (320)$$

According to the Trotter product formula higher order schemes can be obtained if there is a correction term E such that the replacement of $(B^{r,ij}(t_m))$ by $(B^{r,ij}(t_m)) + E$ cancels the correction of order q at each time step which may be defined by

$$\begin{aligned} C^q(t_r)_{\text{corr}} &= C^q(t_r) - (\delta_{ij}D) + (B^{r,ij}(t_r)) \\ &= \sum_{p=1}^q \frac{1}{p!} \sum_{l \geq 1, \beta^l \in \mathbb{N}_{10}^l, m' + |\beta^l| = p, l \leq m'+1} (\delta_{ij}D)^{m'} \times \\ &\times I_{\beta^{m'}} [\Delta(B^{r,ij}(t_r)), (B^{r,ij}(t_r))]_T. \end{aligned} \quad (321)$$

The explicit analysis of these schemes has a right in its own and will be considered elsewhere. For simulations we have to make a cut-off leaving only finitely many modes, of course. The formulas established remain true if we consider natural projections to systems of finite modes. We define

$$\begin{aligned} &(\exp\left(\frac{t}{m}P_{M^l}(\delta_{ij}D)\right) \exp\left(\frac{t}{k}P_{M^l}(B^{r,ij}(t_m))\right)) \times \\ &\cdots (\exp\left(\frac{t}{m}P_{M^l}(\delta_{ij}D)\right) \exp\left(\frac{t}{k}P_{M^l}(B^{r,ij}(t_0))\right)) P_{v^l} \mathbf{h}^F, \end{aligned} \quad (322)$$

where the operator P_{v^l} is defined by

$$P_{v^l} \mathbf{h}^F = (P_{1v^l} \mathbf{h}_1^F, \dots, P_{nv^l} \mathbf{h}_n^F) \quad (323)$$

such that for $1 \leq i \leq n$ and $h_i = (h_{i\alpha})_{\alpha \in \mathbb{Z}^n}$ we define

$$P_{iv^l}(v_\alpha)_{\alpha \in \mathbb{Z}^n} = \left(v_{i\alpha}^l\right)_{|\alpha| \leq k}, \quad (324)$$

and for infinite matrices $M = (M^{ij}) = \left(\left(m_{\alpha\beta}^{ij}\right)_{\alpha, \beta \in \mathbb{Z}^n}\right)$ we define

$$P_{M^l} M = (P_{ij, M^l} M^{ij}), \quad (325)$$

where

$$P_{ij,M^l} M^{ij} = \left(m_{\alpha\beta}^{ij} \right)_{|\alpha|,|\beta| \leq k} \quad (326)$$

In both cases (vector- and matrix-projection) we understand that the order of the modes is preserved of course. Next it is a consequence of our existence constructive proof that

Theorem 4.1. *Let $h_i \in C^\infty(\mathbb{T}_l)$, and let $T > 0$ be a time horizon for the incompressible Navier-Stokes equation problem on the n -torus. Let $s \geq n+2$. Then the finite mode scheme defined in (322) converges for each $0 \leq t \leq T$ to the solution of incompressible Navier-Stokes equation in its infinite nonlinear ODE representation for the Fourier modes as $m, l \uparrow \infty$.*

Next recall that the parameter $\nu > 0$ can be chosen arbitrarily. A large parameter $\nu > 0$ increases damping and makes the computation more stable. However, in order to approximate a Cauchy problem (the initial value problem on the whole space) we need large l (length of the torus). Note that the damping terms in the computation scheme are of order $\frac{1}{l^2}$, the convection terms are of order $\frac{1}{l}$ and the Leray projection terms are independent of the length of the torus. They become dominant if the size l of the n -torus is large. From our constructive existence proof we have

Corollary 4.2. *The Cauchy problem for the incompressible Navier Stokes equation may be approximated by a computational approximation of order k on the n -torus \mathbb{T}_l for l large enough where in a bi-parameter transformation of the original problem $\nu > 0$ should be chosen*

$$\nu \gtrsim l^2 \quad (327)$$

such that the damping term is not dominated by the Leray projection terms. Furthermore due to the quadratic growth of the moduli of diagonal (damping) terms with the order of the modes compared to linear growth of the convection term and the Leray projection term with the order of the modes the scheme remains stable for larger maximal order k if it is stable for lower maximal order k . Furthermore the choice

$$\nu \geq 2|h|_s^2 \quad (328)$$

for $s \geq 2 + n$ leads to solutions which are uniformly bounded with respect to time.

For numerical and computational purposes it is useful to represent initial data in cos and sin terms (otherwise computational errors may lead to nonzero imaginary parts of the computational approximations). We have

Lemma 4.3. *Functions on the n -torus \mathbb{T}_l^n be represented by symmetric data, i.e. with the basis of even functions*

$$\left\{ \cos \left(\frac{2\pi \alpha x}{l} \right) \right\}_{\alpha \in \mathbb{Z}^n}. \quad (329)$$

The reason for the statement of lemma 4.3 simply is that you may consider a L^2 - function f on the cube $[-l, l]^n$ (arbitrary prescribed on $[0, l]^n$) with periodic boundary conditions with respect to a general basis for $L^2(\mathbb{T}_l^n)$ such that

$$f(x) = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \exp\left(\frac{2\pi i \alpha x}{l}\right), \quad (330)$$

and then you observe that $f(\cdots, x_i, \cdots) = f(\cdots, -x_i, \cdots)$ imply that the sin-terms implicit in the representation (330) cancel. Hence on the n -torus which is built from the cube $[0, l]^n$ we may consider symmetric data without loss of generalization.

5 The concept of turbulence

In the context of the results above let us make some final remarks concerning the concept of turbulence. There are several authors (cf. [2] and [3] for example), who emphasized that the dynamical properties of the Navier-Stokes equation such as the existence of strange attractors etc. rather than the question of global smooth existence may be important for the concept of turbulence. We share this view, and we want to emphasize from our point of view, why we consider the infinite dynamical systems of velocity modes to be the interesting object in order to start the study of turbulence. This concept is indeed difficult, and in a perspective of modelling it is always worth to follow this difficulty up to its origins on a logical or at least very elementary level. According to classical (pre-Fregean) logic, a concept is a list of notions, each of which is a list of notions and so on. This is not a definition but something to start with (the definition has some circularity as there is no sharp distinction between a concept and a notion, but this may be part of an inherent difficulty to define the concept of concept). According to Leibniz a concept is clear (german: 'klar') if enough notions are known in order to decide the subsumption of a given object under a concept, and a concept is conspicuous (german: 'deutlich') if there is a complete list of notions of that concept. All this is relative to a 'cognoscens' of course. Since Leibniz expected that even for empirical concepts like 'gold' the list of notions may be infinite he expected a conspicuous cognition to be accessible only to an infinite mind and not to human beings. Anyway, according to this concept of concept we may say that our concept of the concept of turbulence may not even be clear. However, there is some agreement that some specific notions belong to the notions of 'turbulence'. Some aspects of fluid dynamics may be better discussed with respect to more advanced modelling. For example for a realistic discussion of a 'whirl' we may better define a Navier-Stokes equation with a free boundary in one (upper) direction and a gravitational force in the opposite direction (pointing downwards). This free boundary may be analyzed by front fixing on a fixed half space as we did it in the

case of an American derivative free boundary problem. Additional effects are created by boundary conditions (the shape of a river bed, for example). Indeed without boundary conditions dissipative features will dominate in the end. First we observe that it is indeed essential to study the dynamics of the modes. The Navier-Stokes equation is a model of classical physics ($|\mathbf{v}| \leq c$, where $c > 0$ denotes the speed of light. Therefore physics should be invariant with respect to the Galilei transformation. If $\mathbf{v} = (v_i)_{1 \leq i \leq n}^T$ is a solution of the Navier Stokes equation then for a particle which is at $\mathbf{x}_0 = (x_{01}, \dots, x_{0n})$ at time t_0 we have the trajectory $x(t)$, $t \geq t_0$ determined by the equation

$$\dot{\mathbf{x}}(t) = \mathbf{v}, \text{ i.e., } x_i(t) = x_{0i} + \int_{t_0}^t v_i(s) ds, \quad 1 \leq i \leq n, \quad (331)$$

which means that for the modes we have

$$x_{\alpha i} + \int_{t_0}^t v_{i\alpha}(s) ds, \quad 1 \leq i \leq n, \quad (332)$$

Hence, rotationality, periodic behavior, or irregular dynamic behaviour due to a superposition of different modes $v_{i\alpha}$ which are 'out of phase' translates to an analogous dynamic behavior for the trajectories with a superposition of a uniform translative movement (which disappears in a moved laboratory and is physically irrelevant therefore. up to Galiliei transformation dynamic behavior of the modes translates into equivalent. The dynamics of the modes is an infinite ODE which may be studied via bifurcation theory. Since we proved the polynomial decay of modes a natural question is wether a center manifold theorem and the existence of Hopf bifurcations may be proved. The quadratic terms of the modes in the ODE of the incompressible Navier Stokes equation in the dual formulation seem to imply this. Irregular behavior may also be due to several Hopfbifurcations with different periodicity. Bifurcation analysis may lead then to a proof of structural instability, sensitive behaviour and chaos (cf. [1]). Global sensitive behavior and chaos may also be proved via generalisations or Sarkovskii's theorem or via intersection theory of algebraic varieties as proposed in [12] and [13]. Studying the effects of boundary conditions and free boundary conditions for turbulent behavior may also be crucial. For the Cauchy problem dissipative effects will take over in the long run. For a 'river' modelled by fixed boundary conditions for the second and third velocity components v_2 and v_3 (river bank) and a free boundary condition (surface) and a fixed boundary condition (bottom) for the first velocity component v_1 , and a gravitational force in the direction related to the first velocity component, complex dynamical behavior can be due to the boundary effects. It is natural to study such a boundary model by the front fixing method considered in [5]. It is likely that the scheme considered in [7] and [8] can be applied in order to obtain global existence

results.

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